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Systoles of hyperbolic 3-manifolds

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1. *Introduction*

Let M be a complete hyperbolic n-manifold of finite volume. By a *systole* of M we mean a shortest closed geodesic in M. By the *systole length* of M we mean the length of a systole. We denote this by $\mathrm{sl}\,(M)$. In the case when M is closed, the systole length is simply twice the injectivity radius of M. In the presence of cusps, injectivity radius becomes arbitrarily small and it is for this reason we use the language of 'systole length'.

In the context of hyperbolic surfaces of finite volume, much work has been done on systoles; we refer the reader to [**2**, **10**–**12**] for some results. In dimension 3, little seems known about systoles. The main result in this paper is the following (see below for definitions):

Theorem 1·1. *Let M be a closed orientable* 3-*manifold which does not admit any Riemannian metric of negative curvature. Let L be a link in M whose complement admits a complete hyperbolic structure of finite volume. Then* $\mathrm{sl}\,(M\backslash L) \leqslant 7{\cdot}35534\ldots$.

A particular case of this is when $M = S^3$. We define the systole length of a link L with hyperbolic complement to be that of $S^3\backslash L$.

Corollary 1·2. *Let $L \subset S^3$ be a knot or link with hyperbolic complement. Then* $\mathrm{sl}\,(L) \leqslant 7{\cdot}35534\ldots$.

If we specify the type of link, we can sometimes do better (see Theorem 3·4).

We also utilize known bounds on systole lengths of hyperbolic surfaces to obtain bounds on systole lengths for hyperbolic 3-manifolds containing such surfaces. For example, we prove in Section 5.

Theorem 5·1. *Let M be a compact* 3-*manifold with nonempty torus boundaries and hyperbolic interior, such that it contains a properly embedded or properly immersed incompressible boundary-incompressible orientable surface $S_{g,p}$, with genus g and p punctures, $p \geqslant 1$, that has no accidental parabolics. Then* $\mathrm{sl}\,(M) \leqslant \mathrm{sl}\,(S_{g,p})$ *where*:

(i) *if* $p = 1$, $\mathrm{sl}\,(S_{g,p}) \leqslant 2\,\mathrm{arccosh}\,((6g-3)/2)$;

(ii) *if* $p \geqslant 2$ *and* $(g,p) \neq (0,3)$, *then* $\mathrm{sl}(S_{g,p}) \leqslant 2\,\mathrm{arccosh}\,((12g+5p+13)/2)$;

(iii) *if* $p \geqslant 2$ *and* $(g,p) \neq (0,3)$, *then* $\mathrm{sl}\,(S_{g,p}) \leqslant 4\,\mathrm{arccosh}\,((6g-6+3p)/p)$.

We remark that the constraint that the surface have no accidental parabolics can be dropped (see Theorem 5·2). In addition, as a special case of interest we deduce:

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Corollary 5·3. *A hyperbolic* 3-*manifold* M *containing an incompressible boundary-incompressible planar surface has* $\mathrm{sl}\,(M) \leqslant 7{\cdot}05098\ldots$.

Note that Thurston's Hyperbolic Dehn Surgery Theorem [**13**], implies that for all but a finite number of Dehn fillings on the hyperbolic complement of a knot or link in a 3-manifold, the systole length will decrease. However, it is not the case that the systole length always decreases, even if we restrict to Dehn filling a knot in the 3-sphere. For example, the 6_1-knot has systole length $0{\cdot}3306\ldots$ while the manifold resulting from $(-1, 1)$-surgery on the 6_1-knot has systole length $0{\cdot}3661\ldots$. Other results on bounding injectivity radii in classes of hyperbolic 3-manifolds are given in [**7**].

2. *Preliminaries*

By a *hyperbolic* 3-*manifold* we shall always mean a complete orientable hyperbolic 3-manifold of finite volume. In this section we recall some basic facts that we shall require (see [**4**, **9**] for details).

We shall always work with the upper-half space model of hyperbolic 3-space, so that any hyperbolic 3-manifold is obtained as the quotient $\mathbf{H}^3/\Gamma$ where Γ is a torsion-free Kleinian group, acting on $\mathbf{H}^3$ with a fundamental polyhedron of finite volume.

A *horosphere* $\mathscr{H}$ in $\mathbf{H}^3$, is defined to be the intersection in $\mathbf{H}^3$ of a Euclidean sphere in $\mathbf{H}^3 \cup S^2_\infty$ tangent to S^2_∞ at $p \in S^2_\infty$. The point p is referred to as the *centre* of $\mathscr{H}$. The interior of a horosphere is a *horoball*. When p is the point at ∞, a horosphere is just a horizontal plane at some height t up the z-axis.

Let $M = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold, which is non-compact, but of finite volume. The ends of M consist of a finite number of *cusps* each of which is homeomorphic to $T^2 \times [0, \infty)$. On lifting a cusp C to $\mathbf{H}^3$ we see a disjoint collection of horoballs equivalent under the action of Γ. The projection of a collection of disjoint horoballs covering C in $\mathbf{H}^3$ to M is called a *horoball neighbourhood* of C. We remind the reader that the centres of all the horospheres arising as lifts of all cusps, are precisely the fixed points of parabolic elements in Γ.

Fix one cusp C and consider the collection of disjoint horoballs in the pre-image of C. We can expand these horoballs equivariantly until two first touch. The projection of such a configuration to M is referred to as a *maximal cusp* of M. A cross-section of any cusp is a torus, which we call a *cusp torus*. The *waist size* of a cusp is the length of the shortest essential simple closed curve corresponding to a parabolic isometry on a maximal cusp torus associated to that cusp.

We remark that in the context of manifolds with more than one cusp we can also consider the projection of the configuration in $\mathbf{H}^3$ where any two horoballs first become tangent, not necessarily projecting to the same cusp.

By a framing of a cusp torus T we shall mean a choice of two generators for $\pi_1(T)$. By a *slope* on a cusp torus T we mean an isotopy class of an unoriented simple closed curve on T. If M has n cusps, the result of Dehn filling some or all these cusps along a collection of slopes $(r_1, \ldots, r_n)$ will be denoted $M(r_1, \ldots, r_n)$.

We complete this subsection with the statement of the 2π-Theorem of Gromov and Thurston (see [**5**, **6**]). Recall that the hyperbolic metric restricts to a Euclidean metric on a cusp torus.

Theorem 2·1. *Let* M *be a cusped hyperbolic* 3-*manifold with* n *cusps. Let* $T_1, \ldots, T_n$ *be* *disjoint cusp tori for the n cusps of M, and r_i a slope on T_i represented by a geodesic α_i whose length in the Euclidean metric on T_i is greater than 2π, for each $i = 1, \ldots, n$. Then $M(r_1, \ldots, r_n)$ admits a metric of negative curvature.*

It will be convenient to fix a particular normalization for a maximal cusp. In what follows we shall always ensure that two parabolic fix-points are at 0 and ∞ and that a maximal cusp is arranged so that the point of tangency between horoballs centred at ∞ and 0 occurs at height 1. We will call this the *standard form* for a maximal cusp. In addition, in what follows the horosphere arising as the Euclidean plane $z = 1$ will be denoted $\mathscr{H}$. All horoballs tangent to $\mathscr{H}$ in this configuration will be called *full-sized horoballs.*

It is an elementary consequence of the definition of the hyperbolic metric on $\mathbf{H}^3$ that Euclidean distance and hyperbolic distance are the same on $\mathscr{H}$. We will make use of the following lemma that is a consequence of Theorem 2·1 above.

LEMMA 2·2. *Let M be a closed oriented 3-manifold which does not admit any Riemannian metric of negative curvature. Let $K \subset M$ be a knot whose complement admits a complete hyperbolic structure of finite volume. Let (μ, ℓ) be a framing for the cusp torus of $M \backslash K$ in which μ is a meridian. Then the length of μ in a maximal cusp in standard form is no greater than 2π.*

Proof. By assumption M admits no metric of negative curvature, hence the 2π-Theorem implies that for a maximal cusp in standard form the length of μ is at most 2π.

We remark that for a maximal cusp in standard form, since all horospheres are equivalent to $\mathscr{H}$ under the action of the fundamental group, which is by isometries, all conjugates of μ must have length at most 2π measured on an appropriate horosphere.

3. *Main results for knots*

As it contains the main idea we prove the following version of Theorem 1·1 to begin with. Links will be dealt with in Section 4 below.

THEOREM 3·1. *Let M be closed orientable 3-manifold which does not admit any Riemannian metric of negative curvature and $K \subset M$ be a knot with hyperbolic complement. Then* $\mathrm{sl}\,(M \backslash K) \leqslant 4\pi$.

Proof. Let $\Gamma = \pi_1(M \backslash K)$ with a specified basepoint $*$ being the point of tangency of a maximal cusp in $M \backslash K$. Abusing notation slightly, we also identify Γ with a subgroup of $\mathrm{SL}\,(2, \mathbf{C})$. Lifting to $\mathbf{H}^3$, we arrange so a maximal cusp is in standard form and a pre-image of $*$ is the point $p = (0, 0, 1)$. Let x_1 be a lift of a meridian of K (based at p) lying on $\mathscr{H}$ and x_2 (also based at p) a lift of a meridian to the horosphere bounding the full-sized horoball centred at 0. The path $x_1^{-1} x_2$ projects to a loop g in $M \backslash K$. If this path determines a loxodromic element, Lemma 2·2 and the remark following it imply that g has length at most 4π, hence the unique geodesic in the free homotopy class of g has length bounded by 4π. This proves the Theorem, unless $x_1^{-1}x_2$ projects to a loop which is parabolic. However in this case, it is easy to see (as we establish below) that $\langle x_1, x_2\rangle$ is conjugate in $\mathrm{SL}(2,\mathbf{C})$ to the group

$$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle,$$

which is the level 2 congruence subgroup in the modular group, and hence $\langle x_1, x_2\rangle$ gives rise to a twice-punctured disc in $M\backslash K$. The unique hyperbolic structure on a twice punctured disc has shortest closed geodesic of length $2\ln(3+2\sqrt{2}) = 3{\cdot}525494\ldots$. This will complete the proof.

We show here that $\langle x_1, x_2\rangle$ is conjugate to

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Without loss of generality we can conjugate so that

$$x_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad\text{and}\quad x_2 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}.$$

Since the product $x_1^{-1}x_2$ is parabolic, $\mathrm{tr}(x_1^{-1}x_2) = \pm 2$. The trace cannot equal 2 as this would force x_2 to be the identity. Hence the trace is -2 and so we deduce that $r = 2$ as required.

The following theorem, of interest in its own right, allows us to reduce the bound of 4π.

THEOREM 3·2. *Let N be a finite volume hyperbolic 3-manifold with at least one cusp. Assume that in a maximal cusp torus, there is a non-trivial curve corresponding to a parabolic isometry of length equal to w. Then*:

(1) $\mathrm{sl}(N) \leqslant \mathrm{Re}\, 2 \operatorname{arccosh}((2+w^2 i)/2)$ *if* $w \neq 2$;

(2) $\mathrm{sl}(N) \leqslant 2\ln(3+2\sqrt{2}) = 3{\cdot}525\ldots$ *if* $w = 2$.

Proof. As in Theorem 3·1 we consider a specified basepoint $*$ being the point of tangency of a maximal cusp in N, and for which on lifting $\mathbf{H}^3$, we arrange so a maximal cusp is in standard form and a pre-image of $*$ is the point $p = (0,0,1)$. Let x_1 be the lift of the non-trivial curve c on the cusp boundary to the horosphere centred at ∞ with beginning endpoint at p and final endpoint at p', which has coordinates $(w, 0, 1)$. There is a lift x_2 of a non-trivial curve parallel to c on the cusp boundary with the same basepoint, the lift lying on the horosphere $\mathscr{H}_1$ bounding the full-sized horoball centred at $(0,0,0)$, having end-points p and some point α. Note that the product of the corresponding parabolics is itself parabolic only in the specific case that $w = 2$ and x_1 and x_2 make an angle of 0. We first exclude that case from consideration.

By replacing x_2 by x_2^{-1} we may assume without loss that the angle between x_1 and x_2 at p is at most $\pi/2$. The greatest length of a geodesic corresponding to the loxodromic isometry that is the product of the corresponding pair of parabolic isometries occurs when the two paths do meet at right angles. Hence, we look at the isometry corresponding to $x_1^{-1}x_2$ when x_1 and x_2 are at right angles. We can represent the parabolics corresponding to x_1^{-1} and x_2 by

$$\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ iw & 1 \end{pmatrix}.$$

The trace of the product is $2+iw^2$. The result (1) then follows from the standard identity relating complex length and trace.

When $w = 2$, it could be the case that x_1 and x_2 do meet at angle 0, however, then they generate a thrice-punctured sphere in the manifold, and moreover, $x_1^{-1}x_2^{-1}$ will be loxodromic of length $2\ln(3+2\sqrt{2}) = 3{\cdot}525\ldots$. Since this exceeds the bound we have from inequality (1) when $w = 2$, this becomes our upper bound in this case.

In the case of knots in manifolds admitting no metric of negative curvature, so that $w \leqslant 2\pi$, we have:

COROLLARY 3·3. *Let M be closed orientable 3-manifold which does not admit any Riemannian metric of negative curvature and $K \subset M$ be a knot with hyperbolic complement. Then* $\mathrm{sl}(M\backslash K) \leqslant 7{\cdot}35534\ldots$.

THEOREM 3·4. *Let $L \subset S^3$ be a 2-bridge knot or link with hyperbolic complement. Then* $\mathrm{sl}(L) \leqslant 2\,\mathrm{arccosh}(1+\sqrt{2}) = 3{\cdot}057\ldots$. *Moreover, the systole length is realized by a simple geodesic.*

Proof. In [**1**], it was shown that if K is a 2-bridge knot with hyperbolic complement (meaning it is not a 2-braid), then the waist size of the knot is strictly less than 2. Hence w as in Theorem 3·2 can be assumed to be less than 2, which gives the result. If L is a link with more than one component, it has two components and it was shown in [**1**] that there is a choice of cusps such that either one touches itself or two touch each other, and each of those cusps contains a non-trivial curve of equal length less than 2. In the first case, we can apply Theorem 3·2 directly. In the second case, we apply the proof of the theorem to the two parabolics that wrap around the two distinct cusps and that share a basepoint at the point of tangency of the two cusps.

If the shortest geodesic in the manifold were not simple, then if it is thought of as the product ab in the fundamental group, by cut-and-paste, it is clear that a, b and ab^{-1} must have shorter representative loops and therefore each be parabolic. However, this implies that all these elements lie in the fundamental group of an incompressible, boundary-incompressible thrice-punctured sphere immersed or embedded in the manifold. In particular, the shortest geodesic must have length $3{\cdot}525\ldots$, contradicting our previous upper bound on that length.

Remark. Notice that the last paragraph of the proof of Theorem 3·4 shows that for any finite volume hyperbolic 3-manifold M the shortest geodesic is simple unless $\mathrm{sl}(M) = 3{\cdot}525\ldots$.

4. *Links*

We now prove Theorem 1·1, which we state again here for convenience.

THEOREM 1·1. *Let M be a closed orientable 3-manifold which does not admit any Riemannian metric of negative curvature. Let L be a link in M whose complement admits a complete hyperbolic structure of finite volume. Then* $\mathrm{sl}(M\backslash L) \leqslant 7{\cdot}35534\ldots$.

Proof. Theorem 2·1 implies that for any collection of disjoint cusps, there must be at least one with shortest non-trivial curve of length less than 2π. Thus, choose any collection of disjoint cusps in $M\backslash L$. Suppose first that there is exactly one cusp in the set such that its shortest non-trivial curve is of length at most 2π. Begin expanding that cusp. If it expands all the way to maximal, while its waist remains at most 2π, then once it is maximal, it is tangent to itself and we can apply the argument as in Theorems 3·1 and 3·2 to get the bound stated.

Otherwise, at some point, the waist size of the cusp becomes greater than 2π as we expand it. However, then it must have bumped into another cusp and forced it to shrink as it expands, so that there will still be a cusp in the set with waist size at most 2π. In this case, there must have been a point where both of these cusps touched and both had equal waist size at most 2π. At this point, take basepoint at the point of tangency and argue as before taking the waist of one times waist of other.

Now suppose that there are two or more cusps with waist sizes at most 2π. Choose one and expand it. If it keeps waist size at most 2π until it is maximal, then we are done. Otherwise, if its waist gets larger than 2π, and it does not force any other cusps to have waist size below 2π, we are in the case of one fewer cusps with waist size below 2π and we repeat the process. On the other hand, if as we expand this cusp, it does shrink another cusp to waist size at most 2π, before it goes to waist size greater than 2π, there must have been a point where both of these cusps were touching and had equal waist size at most 2π. So again we may take the basepoint at the point of tangency and take the product of their waists, to obtain the required bound for the length of the systole.

Remark. Note that it is an elementary consequence of residual finiteness of Kleinian groups that there are hyperbolic 3-manifolds of finite volume with arbitrarily large systole lengths. As every closed 3-manifold is obtained by surgery on a hyperbolic link in S^3 ([**8**]), this shows that $\mathrm{sl}(M)$ can increase an arbitrarily large amount under Dehn surgery.

5. *Surfaces in 3-manifolds*

Note that if $S_{g,p}$ is a properly embedded or properly immersed incompressible boundary-incompressible orientable surface with genus g and p punctures in a hyperbolic 3-manifold, then $(g,p) \neq (1,0), (0,1)$ or $(0,2)$.

THEOREM 5·1. *Let M be a compact 3-manifold with nonempty torus boundaries and hyperbolic interior, such that it contains a properly embedded or properly immersed orientable incompressible boundary-incompressible surface $S_{g,p}$, $p \geqslant 1$, that has no accidental parabolics. Then* $\mathrm{sl}(M) \leqslant \mathrm{sl}(S_{g,p})$ *where*:

(i) *if* $p=1$, $\mathrm{sl}(S_{g,p}) \leqslant 2\operatorname{arccosh}((6g-3)/2)$;

(ii) *if* $p \geqslant 2$ *and* $(g,p) \neq (0,3)$, *then* $\mathrm{sl}(S_{g,p}) \leqslant 2\operatorname{arccosh}((12g+5p-13)/2)$;

(iii) *if* $p \geqslant 2$ *and* $(g,p) \neq (0,3)$, *then* $\mathrm{sl}(S_{g,p}) \leqslant 4\operatorname{arccosh}((6g-6+3p)/p)$.

Proof. Let $f: S \to M$ be the immersion. Choose an ideal triangulation of S. Each edge in the ideal triangulation is mapped to an edge in M, which then lifts to a set of edges in $\mathbf{H}^3$, with well-defined endpoints on the boundary of $\mathbf{H}^3$. These can be replaced with geodesic edges in $\mathbf{H}^3$ and the triangular faces can be replaced with ideal triangles, which project down to an immersed pleated surface in M made up of the images of the ideal triangles. The local metric on the immersed surface in M pulls back to a particular hyperbolic metric on S. A systole on that surface is mapped to a particular geodesic on the immersed straightened surface. This corresponds to a geodesic in the manifold of no greater length. Thus, $\mathrm{sl}\,(M) \leqslant \mathrm{sl}\,(S)$. The upper bounds for the systole lengths of a surface with genus g and p punctures come from [**2**] for (i) and (ii) and [**11**] for (iii).

Example. If M is a once-punctured torus bundle, it contains an embedded copy of the fibre. Hence, we see that $\mathrm{sl}\,(S^3-K) \leqslant \mathrm{sl}\,(S_{1,1}) = 1{\cdot}9428\ldots$. In fact, for the figure-eight knot complement, which is a once-punctured torus bundle, the fibre becomes an immersed once-punctured torus. The figure-eight knot complement has systole length $1{\cdot}087\ldots$.

THEOREM 5·2. *Let M be a compact 3-manifold with nonempty torus boundaries and hyperbolic interior, such that it contains a properly embedded or properly immersed incompressible boundary-incompressible surface $S_{g,p}$, with genus g and p punctures, $p \geqslant 1$.*

(i) *If* $p \geqslant 2$, *then* $\mathrm{sl}\,(M) \leqslant \max\{3{\cdot}525\ldots, \min\{2\operatorname{arccosh}((12g+5p-13)/2), 4\operatorname{arccosh}((6g-6+3p)/p)\}\}$.

(ii) *If* $p=1$, *then* $\mathrm{sl}\,(M) \leqslant 3{\cdot}525\ldots$ *for* $g=1$ *and* $\mathrm{sl}\,(M) \leqslant 2\operatorname{arccosh}(6g-5)$ *for* $g \geqslant 2$.

Proof. In the case there are no accidental parabolics, the result is immediate. Suppose now that the immersed surface S has an accidental parabolic curve. We will compress the surface to the boundary of the manifold along the accidental parabolic. Suppose first we are in the above situation and we are in the case where formula (i) applies. If the compression does not separate the surface, g becomes $g-1$ and p becomes $p+2$ and this lowers the term in formula (i). Thus we may assume the compression separates the surface into two surfaces. That for one of the resulting surfaces, $2\operatorname{arccosh}((12g+5p-13)/2)$ is lowered is straightforward. It remains to show that for one of these surfaces we lower the value of $4\operatorname{arccosh}((6g-6+3p)/p)$. If $g=0$ or 1 then this is easy to see directly. Thus we assume that $g \geqslant 2$ and that after the accidental parabolic compression we get two surfaces of genus a and $(g-a)$ and with b and $c = p+2-b$ punctures respectively. We can assume that each of b and c is greater than 1, for if either equals 1, the other surface will have to cause a lowering of the systole bound from (i).

Suppose then that for the first of these surfaces the quantity $4\operatorname{arccosh}((6g-6+3p)/p)$ is not lowered, so that:

$$\frac{6g-6+3p}{p} < \frac{6a-6+3b}{b}.$$

Elementary algebra then yields

$$c > \frac{p(g-a)}{(g-1)} + 2. \qquad (*)$$

On the other hand arguing as above for the second surface gives:

$$c < \frac{p(g-a-1)}{(g-1)}. \tag{**}$$

The two equations (∗) and (∗∗) give a contradiction.

In formula (ii), when $p = 1$ and $g = 1$, the accidental parabolic compression could create a thrice-punctured sphere, giving the upper bound in this case. Otherwise, when $p = 1$ and $g \geqslant 2$ a similar argument to the above yields a contradiction.

As an example of the application of this theorem, we deduce

COROLLARY 5·3. *If M contains an immersed incompressible boundary-incompressible planar surface*, $\mathrm{sl}\,(M) \leqslant 7{\cdot}05098\ldots$.

Proof. The bound is obtained from formula (i) above when $g = 0$ and p approaches ∞.

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