

D. Gaitsgory : De Jong's Conjecture

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X/\mathbb{F}_q normally $\rho: \pi_1(X) \rightarrow GL_n F$ $F = \mathbb{F}_q((t))$

$\bar{\rho}$ = restriction of ρ to $\overline{\rho: \pi_1(X) \hookrightarrow \pi_1(X) \rightarrow \hat{\mathbb{Z}} \rightarrow 1}$

Conj $\text{Im } \bar{\rho}$ is finite

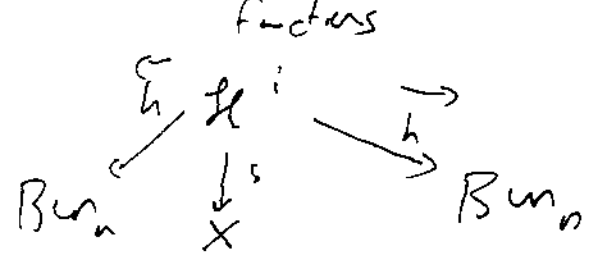
De Jong should suffice to assume X is a complete curve, $\bar{\rho}$ is absolutely irreducible

Bun_n - moduli stack of GL_n bundles on X

De Jong shows his conjecture follows if to ρ can associate f_ρ F -valued function on $\text{Bun}_n/\mathbb{F}_q$ which is Hecke eigenfunction w/ eigenvalues determined by ρ .

Careful! coefficients not in characteristic zero!

$x \in |X|$ H_x^i $i=0, \dots, n$ Hecke operators on automorphic functions



$$\mathcal{H}^i = (x, M, M', M \xrightarrow{\beta} M')$$
 s.t. M'/M is scheme theoretically supported at x & is of length i

$$H_x^i = h_1 \leftarrow h^* (-) \cdot q^{\frac{i(n-i)}{2}}$$

Hecke eigenfunction: $\mathcal{H}_x^i(f_\rho) = \text{Tr}(\rho(\text{Frob}_x)) \cdot f_\rho$

$\rho \leftrightarrow E$ n -dim local system of F -vector spaces on X

Note! q -adic stack is not stack in étale topology of \mathbb{Q}_p -v-spaces.

Similarly have notion of F -sheaves, similar systems of finite coefficient systems

Def $F \in D(\text{Bun}_n)$ (F-derived category) is called a wedge Hecke eigenstate wrt E if

$$H^i(F) \simeq \wedge^i E \otimes F$$

$[H=H']$

Digression F is an eigenstate wrt E if we are given an isomorphism $H(F) \xrightarrow{\sim} E \otimes F$ s.t.

$$(id_X \times H) H(F) \xrightarrow{\sim} E \otimes E \otimes F \Big|_{X \times X \xrightarrow{\Delta} X} \times \text{Bun} \dots$$

is compatible with the Σ_2 -equivariant structure.

Introduce new stack $\begin{array}{ccc} \text{Mod}_n^d & \xrightarrow{\beta} & \text{Bun}_n \\ \downarrow S & & \downarrow \\ X^{(d)} & & X^{(d)} \end{array} = (M, M', \beta: M \rightarrow M'$
 s.t. M'/M is of length d)

S takes data to $\det(M'/M)$.

Locally picture is $\begin{array}{c} \text{cyl } d \\ \downarrow \\ h/W \end{array}$

$E \rightsquigarrow$ Lombar's stack \mathcal{L}_E^d on Mod_n^d :
 $j: X^{(d)} \subset X^{(d)}$ complement of divisors

(reg s-s elements) $\begin{array}{ccc} \text{Mod}_n^d & \xrightarrow{j} & \text{Mod}_n^d \\ \downarrow S & & \downarrow \\ X^{(d)} & \xrightarrow{j} & X^{(d)} \end{array}$ away from divisor Mod_n^d is just d-lines in fibers $\Rightarrow d$ projective spaces \mathbb{P}^{n-1}

So from E define by symmetric product $E^{(d)}$ on $X^{(d)}$

$j_* S^*(E^{(d)}) =: \mathcal{L}_E^d$

Satake equivalence let $\mathcal{H}_X = (M, M', \beta: M \xrightarrow{\sim} M' |_{X=1})$

\Rightarrow F -representations $\text{Rep}(\check{G}_n) \xrightarrow{\sim} \text{Perv}(\mathcal{H}_X)$
(Mirković-Vilonen)

\mathcal{H} $D, M, M': M \xrightarrow{\sim} M' |_{X=1}$

\downarrow
 $D \in X^{(d)}$ [note above stacks had $M \subset M'$: no poles & we fixed degree of quotient \leftrightarrow constant]

More generally:
 $\text{Perv}(\mathcal{H})$ stacks on $X^{(d)}$ with an action of \check{G}_n

$$\text{Perv}_{\check{G}_n} \longrightarrow \text{Perv}(\mathcal{H})$$

V : standard n -dim rep of \check{G}_n (over F)

$$(V \otimes E)^{(d)} \in \text{Perv}_{\check{G}_n} \longrightarrow \text{Perv}(\mathcal{H})$$

goes to Laman's stack

Note: char p must be coeff-defining symmetric power! safest approach: middle extension from compact of divisors.

Laman's stack $\xrightarrow{\text{series of Fourier transforms}} F_E \in \mathcal{D}(\text{Bun}_n)$

moduli obstruction: vanishing result.

automorphic Eisenstein

Obstruction

For $1 \leq n' \leq n-1 \Rightarrow$ averaging factor

$$A_{E'}^d: \mathcal{D}(\text{Bun}_{n'}) \longrightarrow \mathcal{D}(\text{Bun}_n)$$

Need vanishing of $A_{E'}^d$ identically for $d \gg 0$.

Change notations: $\text{rk } E = m > n$, work on Bun_n (which was n' before).