I. Grojnowski : Satake for Double Loop Groups 10/27/04

<u>Theorem</u> (w/ C. Teleman) Let $\hat{\mathcal{B}}$ be the "double loop Grassmannian". Then there is a category of perverse sheaves $\mathcal{P}$ on $\hat{\mathcal{B}}$ & a functor to vector spaces ("deRham cohomology") $\mathbb{H} : \mathcal{P} \to Vect$ s.t.

- For $\mathcal{A} \in \mathcal{P}$, $\mathbb{H}(\mathcal{A})$ is an integrable representation of ${}^L\hat{\mathfrak{g}}$

Moreover there are two convolution structures on $\mathcal{P}$

- First: finite & braided $\xrightarrow{\mathbb{H}}$ fusion tensor product
- Second: not finite but symmetric $\xrightarrow{\mathbb{H}}$ usual tensor product

To prove:
- need guide to phenomena, so not to get lost
- technology for homotopy theory in alg. geometry, for very big spaces
- some hard local geometry

1. Lusztig's Satake isomorphism (after S. Kato, R. Brylinski)
(work on dual side here, switch $G, G^\vee$).

$${}^L\hat{Gr} = {}^LG((z)) / {}^LG[[z]]$$

i Orbits of ${}^LG[[z]]$ $\longleftrightarrow$ X^+ dominant weights

closure relation: $\mu \leq \lambda \iff \lambda - \mu \in \bigoplus_{\alpha \in \Phi^+} \mathbb{N}\alpha$

ii Kazhdan-Lusztig property: pointwise pure. $\Rightarrow$ local IC Poincaré polys

$$K^{IC}_{\mu\lambda} = \Sigma (-1)^i \dim \mathcal{H}^i \; i_\mu^* IC_\lambda \; t^{i/2} \in \mathbb{Z}[t, t^{-1}]$$

iii) Define Hall-Littlewood polynomial

$$P_\lambda = \frac{1}{W_\lambda(t)} \sum_{w \in W} w \left(e^\lambda \prod_{\alpha \in \Phi^+} \frac{1 - te^{-\alpha}}{1 - e^{-\alpha}} \right)$$

$W_\lambda(t)$ = Poincaré polynomial of stabilizer group

Fishel-G.-Teleman: Strong Macdonald conjecture ⋮

χ_λ Weyl character

$$\chi_\lambda = \sum_{\mu \leq \lambda} K_{\mu\lambda} P_\mu \quad \text{change of basis}$$

where $K_{\mu\lambda} = K^{IC}_{\mu\lambda}$ J : poly nomials

- $K_{\mu\lambda} \neq 0 \implies \mu \leq \lambda$, $K_{\mu\mu} = 1$

  If $\mu < \lambda$, $K_{\mu\lambda}(0) = 0$

- $K_{\mu\lambda} \in \mathbb{N}[t]$

$$P_\lambda(1) = \sum_{w \in W\lambda} e^{\lambda}$$
"monomial symmetric function"

$K_{\mu\lambda}(1) = \dim L(\lambda)_\mu$
$\Downarrow$
$K_{\mu\lambda}$ is t-analog of weight multiplicities

Examples 1. $P_0 = 1$

2. $K_{0\theta} = \sum t^{d_i}$ adjoint representation.

2. Combinatorics : $\mathfrak{g}$ Kac-Moody Lie algebra (symmetrizable)

$\rightsquigarrow (\mathfrak{h}, X, \underbrace{\Pi}_{\{\alpha_i\} \in \mathfrak{h}^*}, \underbrace{\Pi^\vee}_{= \{h_i\} \subseteq \mathfrak{h}})$ X lattice in $\mathfrak{h}^*$ containing $\sum \mathbb{Z}\alpha_i$

$\dim \mathfrak{h} - \# I = \# I - \mathrm{rk}$ (Cartan matrix)

t-analog of Weyl denominator
$$\Delta_t = \prod_{\alpha \in \Phi^+} (1 - t e^{-\alpha})^{\dim \mathfrak{g}_\alpha}$$

$\Delta = \Delta_1$ X^+ dominant weights _ X^0 imaginary roots

Definition Hall-Littlewood polynomial
$$P_\lambda = W_\lambda(t)^{-1} \sum_{w \in W} w\left(e^{\lambda} \frac{\Delta_t}{\Delta}\right) \quad \text{(as formal power series)}$$

Proposition I. $\exists$ $a_{\mu\lambda} \in \mathbb{Z}[t]$ polynomials,
s.t.
$$P_\lambda = \sum_{\substack{\mu \leq \lambda \\ \text{dominant}}} a_{\mu\lambda} \chi_\mu, \quad a_{\lambda\lambda} = 1$$

where $\chi_\lambda = P_\lambda|_{t=0}$ Weyl character of integrable rep.

$P_\lambda \in \operatorname{Rep} G$

e.g. P_0 : if $\mathfrak{g}$ is fin dim
$\mu \in X^+$, $\mu \leq 0 \Rightarrow \mu = 0$

So prop. 1 $\Rightarrow P_0 = 1$

If $\mathfrak{g} = \mathfrak{sl}_2[z, z^{-1}] + \mathbb{C}c + \mathbb{C}d$, $\underset{\in \mathfrak{h}}{\delta} = \alpha_0 + \alpha_1$ imaginary root

$\mathbb{Z}$-torsion of dimensions for double loop group: infinitely many dominant weights less than a given one, due to imaginary weights

<u>Proposition</u> $P_0 = 1 \iff \mathfrak{g}$ has no imaginary roots

Example : $\mathfrak{gl}_\infty$

$$\sum_{j \geq 0} \prod_{i \neq j} \frac{1 - t\, z_i/z_j}{1 - z_i/z_j} = \frac{1}{1-t}$$

<u>Def</u> $K_{\mu\lambda}$ is the inverse matrix, i.e.

$$\chi_\lambda = \sum_{\substack{\mu \leq \lambda \\ \mu \in X^+}} K_{\mu\lambda} P_\mu$$

Hope $K_{\mu\lambda}$ local intersection cohomology on $\overline{\mathcal{B}}$

A <u>KL polynomial</u> is a polynomial satisfying Kazhdan–Lusztig combinatorics, recursion procedure.

Completed character ring :

$$R[X]^\wedge = \varprojlim_k R[X] \big/ R[\alpha \in \Phi^+ : ht\, \alpha \geq k]$$

KL involution $^+$: $\mathbb{Z}[t, t^{-1}][X]^\wedge \circlearrowleft$

$( )^+ : e^\lambda \longmapsto e^\lambda \quad t \longmapsto t^{-1}$

$$\Rightarrow \chi^\mu \longmapsto \chi^\mu \ , \quad P_\lambda \longmapsto P_\lambda + \sum_{\substack{\mu < \lambda \\ \mu \in X^+}} \mathbb{Z}[t, t^{-1}] P_\mu$$