

S. Loktev - Current Algebras & Representations 10/27/07
 (w. B. Feigin) Lie algebras of reductive. A assoc algebra w/ unit

Def (Gollub (Chari-Pressley)) \rightarrow $l\text{-dim } M_{\mathbb{C}} \cong \Gamma(M, \mathcal{O})$ M affine
 construct representations of current algebra $\mathfrak{g} \otimes A$
 $\lambda: \mathfrak{b} \rightarrow \mathfrak{h} \rightarrow \mathbb{C}$ $\varepsilon: A \rightarrow \mathbb{C}$ augmentation

Def Weyl module $W_{\lambda, \varepsilon} =$ Maximal fin dim
 $\mathfrak{g} \otimes A$ - module gen. by vector $v_{\lambda, \varepsilon}$
 $(\mathfrak{g} \otimes P) v_{\lambda, \varepsilon} = \lambda(\mathfrak{g}) \varepsilon(P) v_{\lambda, \varepsilon}$
 for $g \in \mathfrak{b}$ $P \in A$

Theorem (Chari-Pressley in smooth $l\text{-dim}$ case)

- $W_{\lambda, \varepsilon}$ exist
- $\mathfrak{g} \otimes A^N$ acts by zero for $N \gg 0$

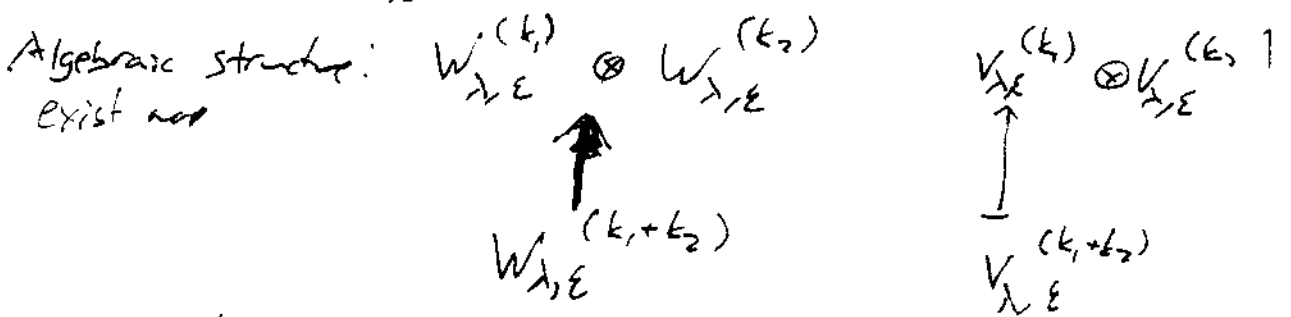
[Maximal: maps to any such module - maximal fin dim quotient of induced module, if it exists]

fin dim:

3. Any $\mathfrak{g} \otimes A$ module generated by a common eigenvector of $\mathfrak{b} \otimes A$ \mathbb{C} -valued currents is a quotient of a product of $W_{\lambda, \varepsilon}$:

Historical Weyl modules:

Definition $W_{\lambda, \varepsilon}^{(k)}$ = submodule of $W_{\lambda, \varepsilon}^{\otimes k}$ generated by $v_{\lambda, \varepsilon}^{\otimes k} = v_{\lambda, \varepsilon}^{(k)}$



-- (co)algebra structure ... coassociative & counital

So get structure of algebra on $\bigoplus_k (W_{\lambda, \epsilon}^{(k)})^*$
 associative & commutative

If $A = \mathbb{C} \Rightarrow$ Functions on Affine Schubert G/B .

Define Schubert variety $Sch_{\lambda, \epsilon} = Proj(\bigoplus_k (W_{\lambda, \epsilon}^{(k)})^*)$

• Dim & character of $W_{\lambda, \epsilon}^{(k)}$?

have an action of \mathfrak{g} & some extras of ring A ,
 preserving $\epsilon \mapsto \epsilon$ i.e. a certain GL (in structure)

- Qs:
1. What is $ch W_{\lambda, \epsilon}^{(k)}$ as $\mathfrak{g} \otimes Aut_{\epsilon} A$ module?
 2. Describe $Sch_{\lambda, \epsilon}$
 3. $Hom_{\mathfrak{g} \otimes A}(W_{\lambda, \epsilon}^{(k_1)}, W_{\lambda, \epsilon}^{(k_2)})$ & limits as $k \rightarrow \infty$...

Examples $A = \mathbb{C}[x^1, \dots, x^d]$ $\epsilon(p) = p(0)$

• $d=0$: $W_{\lambda, \epsilon}^{(k)} = \pi(k\lambda)$ irreducible representation of \mathfrak{g}

Coalgebra is well known. $Sch_{\lambda, \epsilon} =$ image of G/B
 in $\mathbb{P}(\pi(\lambda))$
 No trivial tors ...

• $d=1$ Demazure modules.
 (consider $\mathfrak{g} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c$ central extension
 $L_{k, \lambda}$ integrable representations level k .)

[Statement (known for \mathfrak{sl}_2)]

$v \in L_{k, \lambda}$ highest weight vector. $W(k)$ group acts on $L_{k, \lambda}$
 $v_w = w \cdot v$ $w \in W$

$$v_\alpha = T_\alpha \cdot v \quad \alpha \in T \subset W$$

Denote $D_{\lambda+k\alpha}^{(k)} = U_{\mathfrak{g}} \otimes_{\mathbb{C}[T]} \cdot v_\alpha$

rep of $\mathfrak{g} \otimes_{\mathbb{C}[T]} \mathbb{C} = \hat{\mathfrak{g}}$

Proposition $D_{\lambda+k\alpha}^{(k)}$ is a quotient of $W_{\mu, E}^{(k)}$

... easy for $k=1$, since v_α in this case will satisfy Weyl condition.

μ dominant

Conjecture (Theorem for cyclr) $W_{\mu, E}^{(k)} \cong D_{\lambda+k\alpha}^{(k)}$ as $\mathfrak{g} \otimes_{\mathbb{C}[T]}$ -modules

So Sch are affine Schubert varieties: μ in root lattice get Schubert cell in Grassmannian... $D_\mu \subset Gr$
 otherwise find it in some flag variety $Gr(G)_{\text{aff}}$

Corollary $\mu \in \mathcal{Q}$ $Sch_{\lambda, E} =$ Schubert variety in the affine Grassmann $\mathfrak{g} \otimes_{\mathbb{C}[T]}$ -orbit.

Functions on orbit closure are always Demazure modules
 -- nontrivial statement is that Demazure modules are Weyl, i.e. maximal of this given type.

Singular curve case: \neq nodal case: T. Kuwabara (cyclr)

$d=2$ $\mathfrak{g} = \mathfrak{sl}_2$

Consider noncommutative polynomials

$$\mathbb{C}\langle X, Y \rangle = \langle X, Y \rangle / YX - XY = X$$

(@ natural representation: $X = x$ $Y = x^2/x$
 (want normal ideal to get started --- that's why $YX - XY = X$)

Study $\mathfrak{sl}_2 \otimes_{\mathbb{C}[T]} \mathbb{C}\langle X, Y \rangle$