

Notes by D. Bar-Zvi

Langlands Seminar

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D. Goresen [Beilinson: superscheme $(X, \mathcal{O}_{ev} \oplus \mathcal{O}_{od}^!)$ gives scheme in two different ways: subscheme $(X, \mathcal{O}^*/\langle \mathcal{O}_{od}^! \rangle)$ & quotient (X, \mathcal{O}_{ev}) .]

Derived algebraic geometry: (X, \mathcal{O}_X^*) scheme with sheaf of dg commutative algebra
 Locally X is $\cong \text{Spec } H_*(\mathcal{O}_X^*)$ topologically

Stable homotopy category looks like dg algebras after tensoring with \mathbb{Q} , but very different before tensoring.

Example: X scheme $X_{\bullet}^{alg}(\mathcal{O}_X)$ sheaf of algebraic K -groups $= \pi_0 BGL^+(\mathcal{O}_X, K_0(\mathbb{Z}))$ should pass to homotopy groups only at last stage, work with space for object of stable homotopy category) it's all.

K -theory: algebra (or space) \mapsto spectrum, can be easier to describe than its homotopy groups!

Spectra (pointed spaces, e.g. CW complexes) \rightarrow Spectra in abelian group objects
 \uparrow \uparrow
 Ring spectra \uparrow Ring objects

Abelian group objects? e.g. infinite loop space: sequence of spaces X_n + isomorphism $X_n \xrightarrow{\cong} \Omega X_{n+1}$
 Eilenberg-MacLane spaces $H(A, n), \pi_i H(A, n) = \begin{cases} A & i=n \\ 0 & \text{otherwise} \end{cases}$

$$\pi_{i+1} X \cong \pi_i \Omega X$$

$$\text{Map}(S^{i+1}, X)$$

$$\text{Map}(S^i, \Omega X)$$

FACT The abelian group objects in the homotopy category are products of Eilenberg-MacLane spaces!

Spectra \leftrightarrow \mathcal{S} -modules: modules over sphere spectrum. - analog of abelian groups = \mathbb{Z} -modules in usual world.

Practical

On the other hand a commutative top group is homotopy equivalent to a product of E-M-L spectra.

- - shouldn't think of locally compact spaces, but compactly generated ones - eg realizations of simplicial sets.

Simplicial abelian groups $\xleftrightarrow{\text{Doll. Puppe}}$ chain complexes (non pos graded)
 Each such ^{complex} is (noncanonically) isomorphic to \bigoplus of its cohomology --- ie E-M-L spectra...
 ... since all Ext^i ($i \geq 2$) vanish!
 ... not true more generally

Spectra form symmetric monoidal category under smash...
 What do we mean by a commutative ring spectrum?

naively: rings in this symmetric monoidal category
 Bad idea even in world of complexes!

want dga up to isomorphism, rather than commutative rings in homotopy category of complexes.

Symmetric group action:

$$E\Sigma_n \times_{\Sigma_n} R^1 \dots \wedge R^n \longrightarrow R \quad \text{connectivity data.}$$

Two notions: 1. commutative dga

2. associative dga which are homotopy commutative!

e.g. $ab \simeq ba$ homotopic, + higher compatibilities $\Rightarrow E_{\infty}$ algebras

- same after $\otimes \mathbb{Q}$, different $\neq 0$ characteristic.

Here addition is commutative on the nose.

but could replace this also by something more homotopic \Rightarrow commutative ring spectra

ie spectrum \leftrightarrow weakened version of commutative ring spectrum E_{∞} algebra

$\Rightarrow E_{\infty}$ ring spectrum

Remark $Q(X) := \lim_{n \rightarrow \infty} \Omega^n S^n X \sim \bigcup_{n \geq 0} E_{\Sigma_n} \otimes_{\Sigma_n} X^n / \sim$

$\otimes \mathbb{Q}$: $SP^\infty(X) = V(X^n / \Sigma_n / \sim)$ (bosonics)

Since E_{Σ_n} / Σ_n purely torsion, vanishes $\otimes \mathbb{Q}$.

But homotopy (secretly) very different.

$$\begin{aligned} \Pi_* Q(X) &= \Pi_*^S(X) \quad \dots \text{primitives in homotopy} \\ \text{Hausdorff algebra } H_*(X, \mathbb{Q}) &= H_*(SP^\infty X, \mathbb{Q}) \end{aligned}$$

Enriched scheme (Lurie, Toen-Vezzosi) \sim
 X space, in Zariski topology have sheaf \mathcal{O}_X
of E_∞ ring spectra (comm S -algebra)

Brauer new = functor from old to new.
 X locally homeo to $\text{Spec } \Pi_0 \tilde{\mathcal{O}}_X$.

$S = \text{Brauer new } (\mathbb{Z})$. Good old world \subset brauer new.

$$\mathbb{1} \in \{S^0 \rightarrow H\mathbb{Z}\} = \Pi_0(\mathbb{Z}) = \mathbb{Z} \\ = \tilde{H}^0(S^0, \mathbb{Z})$$

\Rightarrow map $S \rightarrow \mathbb{Z}$ of ring E_∞ ring spectra.

Brauer new-value on \mathbb{Z} is a complex of abelian groups
 $H: D(\mathbb{Z}\text{-mod}) \rightarrow (S^0\text{-Mod})$

$\text{Spec } \mathbb{Z} \xrightarrow{i} \text{Spec } "S"$ enriched scheme
modules over $\text{Spec } S$ is just a spectrum.

$\text{Spec } \mathbb{Z} \rightarrow \text{Spec } S$ far from flat - so can not
via Eilenberg-MacLane, but not in flat way --
any scheme into an enriched scheme