J. Morava II - Enriched schemes, spectral sequences of Adams/ Leray / Descent type, etc... 10/21/04

1. Concrete statements on homotopy groups

Bott: $\pi_* BU = \begin{cases} \mathbb{Z} & * = \text{even} \\ 0 & * = \text{odd} \end{cases}$

$\Leftrightarrow$ $\pi_* U = \begin{cases} \mathbb{Z} & * = \text{odd} \\ 0 & * = \text{even} \end{cases}$ $U = \lim_{n\to\infty} U(n)$

$\mathbb{Z}[\frac{1}{2}] \otimes \pi_* SO = \mathbb{Z}[\frac{1}{2}]$ $* \in 4\mathbb{Z} - 1$
$= 0$ else

(ignore 2-torsion!)
"super" world

"first picoseconds after big bang all force were united"

$\Leftrightarrow$ first few primes all entangled together, things get better for higher primes

Whitehead X space "free" ∞ loop space on X is $\Omega^n S^n X = QX$

$\pi_* QX = \lim_{n\to\infty} \pi_{i+n}(S^n \wedge X) = \pi_*^S(X)$ ---> this is a cohomology theory

$\parallel$

infinite symmetric product on X, rationally

$\exists$ ~~maps~~ $SO(n) \to \Omega^n S^n \Rightarrow J : \pi_* SO \to \pi^S(pt)$

$J : \pi_* SO \longrightarrow \pi_*^S(pt)$ J homomorphism

$\parallel$

$\mathbb{Z}$ $* = 4k-1$ image in dim $4k-1$ is cyclic, ~~of order~~ generated by $b_{2k}/2k \in \mathbb{Q}/\mathbb{Z}$

$b_{2k} \leftrightarrow \zeta(1-2k)$ Bernoulli number.

$S^0 \to E$ unital ring spectrum $\Rightarrow$ Hurewicz map
$\pi_*^S \longrightarrow E_*$

e.g. $E = K_*^{alg}(\mathbb{Z})$: get composition
$$\pi_* SO \longrightarrow \pi_*^S(pt) \xrightarrow[\text{Hurewicz}]{} K_*(\mathbb{Z})$$

composition more or less detects $\zeta(1-2k)$,
[not $K_{4k-1}(\mathbb{Z})$, related to $\zeta(2k)$]

im $J \hookrightarrow K_*(\mathbb{Z})$ in these degrees
$$\downarrow$$
$K_*(\mathbb{F}_\ell)$ computed by Quillen $(\ell, p) = 1$

[Note: Spectra are additive category, Homs $\in$ Ab,
Spectra$_{(p)}$ has Hom in $\mathbb{Z}_{(p)}$-mod]

Work p-locally
Classical fact write $k = 2p^\nu(p-1)\cdot$ ___
Then $\zeta(1-2k) = p^{-(\nu+1)} \in \mathbb{Q}/\mathbb{Z}^{(p)}$ (von Staudt–Clausen)

So p-component of Bernoulli numbers have logic,
can understand p-part of J-homomorphism

2. Adams S Seq, in particular for K-theory

(Spectra)$_{(p)}$ is a triangulated symmetric monoidal category,
with duals for finite objects.

$$H_*(-, \mathbb{F}_p) : (\text{Spectra})_{(p)} \longrightarrow \mathbb{F}_p\text{-Vect} \quad (\text{really } \underline{\text{super}} \text{ graded})$$

1. $\exists$ Künneth theorem: $H_*(X \wedge Y) = H_*(X) \otimes H_*(Y)$
$\Rightarrow$ monoidal functor

... homology of a spectrum: $\lim_{n\to\infty} H_{i+n}(X_n) = H_i(X)$

$=$ primitives in $H^*(\Omega^\infty X) \otimes \mathbb{Q}$

2. Not faithful, But does reflect isomorphisms!
... conservative: If $H_*(f)$ is iso, then
f is a homotopy equivalence.

Consider multiplicative automorphisms of such a functor
$\alpha : H_*(-) \circlearrowleft$ preserving Künneth isomorphism

$\Rightarrow$ $\mathrm{Aut}(H\mathbb{F}_p)$ is a pro-algebraic group,
representing a Hopf algebra, dual to classical
Steenrod algebra of reduced powers
(well also have super part!
$A_* =$ poly $\otimes$ exterior, we're dropping polynomial..
--- have super part)

So can regard $H\mathbb{F}_p : (\mathrm{Spectra})_{(p)} \longrightarrow$ Reps of this pro-alg. group

((Classically $[H\mathbb{F}_p, H\mathbb{F}_p]_* = \mathrm{End}(H\mathbb{F}_p)_* = A^*$)

Adams SSeq:

$$E_2 = \mathrm{Ext}_{\mathrm{Aut}\,H\text{-Reps}}(H_*(X), H_*(Y)) \Rightarrow [X,Y]_{(p)}$$

convergence comes from conservativity of $H\mathbb{F}_p$.

To get SSeq: build resolution of S^0 by $\underline{H\mathbb{F}_p}$-free objects