N. Nekrasov - Physics & higher dim Langlands 10/25/04

"Instanton partition functions"

		pure super YM	SYM w/ adjoint matter	SYM w/ fundamental matter
elliptic cohomology	6d	elliptic cohomology on $\mathcal{M}_{G,k}$	Ell $T^*\mathcal{M}_{G,k}$	ell
K-theory	5d	K on $\mathcal{M}_{G,k}$	K $T^*\mathcal{M}_{G,k}$	K
Cohomology	4d	$H\,\mathcal{M}_{G,k}$ framed sheaves on $\mathbb{P}^2$: cohomology problem	$H\ T^*\mathcal{M}_{G,k}$	H some other bundle

G simple Lie group / $\mathbb{C}$

$$\mathcal{M}_{G,k} = Bun^k_G(\mathbb{P}^2, \mathbb{P}^1_\infty)$$

G-bundles on $\mathbb{P}^2$ trivialized on line at ∞ with $c_2 = k$

k: instanton charge

$$\mathcal{M}_G = \coprod_{k\geq 0}^{\infty} \mathcal{M}_{G,k} \quad \text{"="} \quad \exp\left(\mathbb{C}^2 \times \coprod_{k=0}^{\infty} \mathcal{M}_{G,k}^{\text{centered}}\right)$$

... $Aut(\mathbb{P}^2, \mathbb{P}^1) \circlearrowright \mathcal{M}_G$, in particular translations act, so can consider quotient by translations

.... look like spaces of centered instantons

Diff geometry interpretation: G_c compact group bundles on S^4 with ASD connection

Energy tends to concentrate in some regions $\Rightarrow$ instantons

[Partition fn for $\mathcal{M}_G$ will end up being exp of a "centered" one.]

Pure 5d instanton partition function:

$$Z_5^{inst}[a, \varepsilon_1, \varepsilon_2, \Lambda; \beta] = \sum_{k=0}^{\infty} \Lambda^k \, \underbrace{Tr_{H_{\mathcal{M}_{G,k}}}}_{} \left(e^{\beta a}, e^{\beta\varepsilon_1}, e^{\beta\varepsilon_2}\right)$$

Z_k ||

polynomial fns on $\mathcal{M}_{G,k}$

Fix $T \subset G$, $\mathfrak{t} = \mathrm{Lie}\, T$.

$a \in \mathfrak{t}$ $\quad \varepsilon_1, \varepsilon_2, \Lambda \in \mathbb{C}$

$\mathcal{M}_{G,k}$ acted on by $G \times \mathbb{C}^* \times \mathbb{C}^*$
G acts changing framings, $\mathbb{C}^* \times \mathbb{C}^* \subset \mathrm{Aut}(\mathbb{P}^2; \mathbb{P}^1)$
$e^{\beta\varepsilon_1}, e^{\beta\varepsilon_2} \in \mathbb{C}^* \times \mathbb{C}^*$, $e^{\beta a} \in T \subset G$ framings
Max torus

$G \times \mathbb{C}^* \times \mathbb{C}^* \circlearrowright \mathcal{H}_{\mathcal{M}_{G,k}}$ holomorphic functions on $\mathcal{M}_{G,k}$
Want to describe character

$\mathcal{M}_{G,k}$: Uhlenbeck is affine, complement has high codim, $\mathcal{M}_{G,k}$ quasiaffine: plenty of regular functions - only polys will contribute to trace
... eigenfunctions are polynomials
(note Uhlenbeck not really compactification, only partial: analog to $\mathbb{C} \setminus 0 \to \mathbb{C}$.)

Two origins of noncompactness: $\mathbb{P}^1_\infty$, 2 pointlike instantons
Uhlenbeck deals only with latter --- high codimension
(need this to be normal to know we're calculating right thing from open piece)

Homogeneous pieces of $\mathcal{H}_{\mathcal{M}_{G,k}}$ wrt $G \times \mathbb{C}^* \times \mathbb{C}^*$ are fin dim $\Rightarrow$ character makes sense.

(for KM get dilog identities from asymptotics ???)

Asymptotics of Tr as $\beta \to 0$ (group elt $\to 1$) gives idea of size of space

$\beta \to 0$ $\quad \mathrm{Tr}_{\mathcal{H}}$ $\quad Z_k \sim \frac{1}{\beta^{\dim \mathcal{M}_{G,k}}}$

$\dim \mathcal{M}_{G,k} = 2 \cdot k \cdot h_G^\vee$. So scaling appropriately get limit.

4d : $Z_4^{inst}[a,\varepsilon_1,\varepsilon_2,\Lambda] = \lim_{\beta\to 0} Z_5^{inst}[a,\varepsilon_1,\varepsilon_2;\beta^{2h^\vee}\Lambda,\beta]$

Λ = low energy scale

β = radius of circle $\mathbb{R}^4\times S^1_\beta$ in 5d theory

[Must assume Λ small so series converges & then analytically continue]

Z_4, Z_5 have interesting behavior when $\varepsilon_1,\varepsilon_2 \to 0$:
("exp of spaces" formula)

asymptotics $Z_4^{inst}[a,\varepsilon_1,\varepsilon_2,\Lambda] = \exp\left[\frac{1}{\varepsilon_1\varepsilon_2}F^{inst}(a,\Lambda) + \text{less singular terms}\right]$

(as $\varepsilon_1,\varepsilon_2\to 0$ uniformly)

F^{inst} comes from totally different problem, in geometry of curves...
periodic Toda system

SYM with adjoint matter (adjoint hypermultiplet)
(elliptic CM)

Pure 5d : relativistic Toda (difference Toda: Ruijsenaars/Macdonald

Pure 4d : periodic Toda (for dual to affinization of $\mathfrak{g}$)

~~Pure~~ adjoint 4d : elliptic CM
(7d too high for right amount of SUSY)

adjoint 5d : Λ splits into two parameters m, q

$$Z_5^{inst}[a,\varepsilon_1,\varepsilon_2,m,q;\beta] = \sum_0^\infty q^k \operatorname{Tr}_{H_{\pi T}\mathcal{M}_{G,k}}\left(e^{\beta a}\times e^{\beta\varepsilon_1}\times e^{\beta\varepsilon_2}\times(-e^{\beta m})\right)$$

odd tangent bundle