Toronto
10/04

C. Teleman - Twistings in Gromov-Witten Theory

Project: Topological determination of GW theory of BG, twisted, with descendants

idea from: Freed-Hopkins-T. Verlinde rings

- Woodward-T.: index formulas for bundles on $Bun_G X$
- In families this gives GW invariants for BG
- in nonabelian case have to invert a few things, but have integral candidate — work with K. Feldman

Expected answer: sum of GWs for a point, after coordinate change.

(Givental-Dubrovin conjecture this is case whenever underlying Frobenius algebra is semisimple)

Background

2D TFT: $\bigcirc \longrightarrow V$ vector space

$\sqcup \longrightarrow \otimes$, $\emptyset \longmapsto k$ ground field

pants: $V \otimes V \xrightarrow{m} V$ $\quad$ cap: $V \longrightarrow k$ $\quad$ disk: $1 \in V$

Theorem 2D TFT $\longleftrightarrow$ Frobenius algebra $/k$

In families: family of TFT FTFT

(stronger version) 1) Circle bundle over base $\longmapsto$ (flat) bundle fiber V
by functoriality & $\pi_1(\mathbb{CP}^\infty) = 0 \Longrightarrow$ actually trivial bundle V.

CohFT 2) surface bundle $\Longrightarrow H^*(\text{base}, Hom(V^{\otimes \text{incoming}}, V^{\otimes \text{out}}))$
satisfying some axioms

Instead of H^* could take dga model for

TCFT $<$ $H^*(\text{base}; \mathbb{Q}) \longrightarrow A^*(\text{base})$

KFT — $K^*(\text{base})$ instead

Remark: universal surface bundle lives over moduli $M_{g,n}$ (unmarked circles - just give points)

$\longrightarrow$ suffices to describe when base is $\mathcal{M}_{g,n}$

(use that MCG ~ BDiff)

(weaker) require circles to have basepoint
$\Rightarrow$ boundary circles parametrized,
circle bundles are trivial.

$\Rightarrow$ Universal base is moduli of surfaces with parametrized boundaries, or just circle boundary over $\mathcal{M}_{g,n}$ - unit tangent vectors at each marked point.

<u>Proposition</u> (Idea: folklore, formulation: K. Costello)
Stronger FT extends naturally to the compactified (Deligne-Mumford) spaces $\overline{\mathcal{M}}_{g,n}$
[homotopy quotient of operad of parametrized surfaces by reparametrize is DM operad]

Extension to the boundary: monodromy around nodal degeneration is Dehn twist of cylinder
- nice rotation action if ignore reparametrization

But cylinder map is indep of parametrization of boundary
$\rightarrow$ class on small S^1 around degeneration is not just in $K^*(S^1, V\otimes V)$ but in $K^*_{S^1}(S^1, V\otimes V) = V\otimes V$
- Surface family is constant outside of a cylinder, & on this cylinder can make family S^1 equivariant
$\rightarrow$ since don't need parametrization to define maps make whole picture equivariant $\Rightarrow$ extend across puncture! ☐

Now can write factorization (sewing) axioms for strong RFT / CohFT / KFT etc...

What info do we need?

Observation: If underlying Frobenius algebra is semisimple, $\otimes \mathbb{Q}$ all classes we get are tautological $\Rightarrow$ come from high genus;

genus increasing map  is invertible in semisimple case (get $\mathrm{diag}(\lambda_k^{-1})$, $\lambda_k = \theta(\text{idempotents})$)
— comes from stable classes of cohomology.

Faber Conjecture: Tautological subring of $\overline{\mathcal{M}}_{g,n}$ satisfies Poincaré duality

Potential: encode all integrals of TFT classes against tautological classes

Tautological classes over $\overline{\mathcal{M}}_g$: $\kappa_i = \int_{\substack{\text{universal}\\ \text{curve}}} (\psi = c_1)^{i+1}$

$v_i \in V \Rightarrow$ GW potential

$$\exp\Big\{ \sum_g \hbar^{g-1} \sum_n \int_{\overline{\mathcal{M}}_{g,n}} Z(v_1,\dots,v_n)\, \psi_1^{k_1}\cdots\psi_n^{k_n} \frac{1}{n!} \Big\}$$

If $V = \mathbb{C}$ known (Kontsevich-Witten)

Observe: K-theory [stable] of $\mathcal{M}_g$ is torsion-free
... has analogs of ψ^{i+1} ... index bundle on powers of canonical bundle, & K-theory operations on them, with no extra relation — so hope to get correct integral results..

GW theory: X symplectic compact manifold

$\mathcal{M}_{g,n}(X)$ = compactified moduli of J-hol curves

$V = H^*(X, \mathbb{Q})$

$Z(v_1, \dots v_n) = \int_{\overline{M}_{g,n}(X) \to \overline{M}_{g,n}} E_i^* v_i$

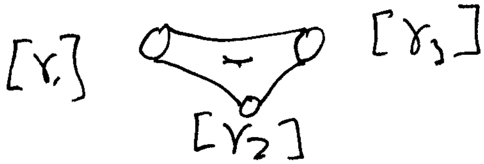
<u>Toy model</u>! (Chen-Ruan) $X = BG$, G finite (Jarvis-Kimura)

---- want $H^*(LBG; \mathbb{Q}) = H^*_G(G, \mathbb{Q}) = \mathbb{C}[G]^G$

Frobenius algebra product is <u>convolution</u> product

---- in usual GW case see $H^*(X)$ since only <u>small loops</u> are relevant, BG case need all loops ...

Maps to BG = flat G-bundles on X, weighted count



count colorings in conjugacy class.

Character basis of $\mathbb{C}[G]^G$: $\chi_\alpha \frac{\deg \alpha}{|G|}$

idempotents $\theta_\alpha = \left(\frac{\deg \alpha}{|G|}\right)^2$

Result! potential $= \sum_\alpha$ (potential for a point) θ_α^{2-2g}

General G compact!

Maps$(\Sigma, BG_{\mathbb{C}})$ = moduli stack of holomorphic $G_{\mathbb{C}}$-bundles

single point: index of trivial bundle $= 1$.

<u>Twisted theories</u>

Twisted GW (Coates-Givental) $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ vector bundle

$$\begin{matrix} E_{n+1}^*(E) \to M_{g,n+1}(X) & \xrightarrow{E_{n+1}} & X \\ \downarrow & & \\ M_{g,n} & & \end{matrix}$$

$\mathrm{Ind}(E^*_{n+1}(E)) \in K^0(M_{g,n}(X))$

~ apply a multiplicative char. class to get to H^*

Twisted GW: insert this characteristic class into GW integrals.

Coates-Givental: "universal" formula for untwisted → twisted.

For BG: $\Sigma \times \mathcal{M}_G(\Sigma) \longrightarrow BG$ classifying map

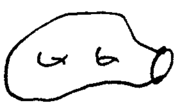
Rep E of $G \Rightarrow$ vector bundle on $BG \Rightarrow$

$\mathrm{Index}_\Sigma(E) \in K^0(\mathcal{M}_G(\Sigma))$

Apply a multiplicative K-theory class

(det, λ_t, S_t, γ_t) $\Rightarrow$

<u>Theorem</u> These indices are computed topologically:



$G^{2g} \longrightarrow G$ product of commutators

$\Pi_* : K_G(G^{2g}) \longrightarrow {}^\tau K_G(G)$

..... exists a twisting of $K_G\, G$ so that $\langle \Pi_* 1, 1 \rangle = \mathrm{Index}$

Works for families of curves (smooth).

What about boundary?

<u>Theorem</u> S^1-equivariant ${}^\tau K_G(G) \simeq R_{S^1} \otimes {}^\tau K_G(G)$

& a choice of splittings gives a strong KTFT

& abelian reduction formula for this index.