Zhu p-adic CFT

A good conformal field theory Π has p-adic completions Π_p s.t. global theory $\Pi_{\mathbb{A}}$ is automorphic.

F local field: $\mathbb{Q}_p$, $\mathbb{R}$, $\mathbb{C}$ ...

$X = F((t)) \times F((t))dt \times F$ Heisenberg group

$a(t)\cdot (b(t)dt) = (b(t)dt)\, a(t)\ \mathrm{Res}\,(a(t)b(t)dt)$

X acts on formal functions $\mathrm{Fun}(F((t))) = \mathrm{Fun}(\ldots a_{-1}, a_0, a_1, \ldots)$ (to be defined)

$(a(t)\cdot f)(x(t)) = f(x(t) + a(t))$ translation

$(b(t)dt \cdot f)(x(t)) = \psi(\mathrm{Res}\ x(t)\, b(t)dt)\, f(x(t))$

$\psi : F \longrightarrow \mathbb{C}^*$ additive character.

C curve$/F$,  $p \in C$ t local parameter

Functional $\phi : \mathrm{Fun}\, F((t)) \longrightarrow \mathbb{C}$ [conformal block ?]

s.t. $\phi(g\cdot f) = \phi(f)$ for $g \in \Gamma(C \setminus p, \mathcal{O})$ or $g \in \Gamma(C\setminus p, \omega)$

-- subgroups of X at p.

$\phi(f)$ "=" $\sum_{x(t) \in F(C-p, \mathcal{O})} f(x(t))$ doesn't make sense ...

Set $D(F((t))) :=$ Span of $\bigotimes_{i=-\infty}^{\infty} f_i$

where f_i is of following types

1. $f_i = \psi_b(a_i) = \psi(b\cdot a_i)$

0. f_i Schwartz function of a_i

-1. $f_i = \delta_b(a_i) = \begin{cases} 1 & a_i = b \\ 0 & a_i \neq b \end{cases}$

allowable distributions

require condition $i << 0$ $f_i = \delta_0$
$i >> 0$ $f_i = \varphi_0 \equiv 1$

-- finite linear span of such functions
in "smooth" part of continuous dual
i.e. part of $\varprojlim S(t^{-m} F[[t]] / t^n F[[t]])$
or ~~~~ Gaitsgory-Kazhdan dual space

$\phi(f)$ should be element of completion ~~of~~ $(D(F((t)))_2)^*$

X acts on $D(F((t)))$, & $\phi(f)$ makes sense on how -summation on sections of line bundles with poles.

~~Our~~ $f_{-i} \equiv \delta_0$ for $i \le -N \Rightarrow$

~~$\phi(f) = \sum_{x(t)\in}$~~

Theorem: There is a unique (up to scalar) continuous
$\phi : D(F((t))) \longrightarrow \mathbb{C}$ which is invariant under
$\Gamma(C-p, \mathcal{O}) \times \Gamma(C-p, \omega)$ ... Lagrangian subgroup (maximal abelian).

For C curve over $\mathbb{Q} \Rightarrow C_v$ curve over every completion $\mathbb{Q}_v$.

Def The space of multiplicative states $(\subset D(\mathbb{Q}((t))))$
~~~~ $\mathcal{G}_v = \{\text{span} \bigotimes_{-\infty}^{\infty} f_i \in D(\mathbb{Q}_v((t))) : f_i \text{ is either type (1) or type (-1)}\}$

Global multiplicative states: $\mathcal{G}(\mathbb{Q}) \hookrightarrow \mathcal{G}_v$ all v

Theorem ~~~~ Suppose C curve over $\mathbb{Q}$
$p \in C(\mathbb{Q})$, t rational local parameter $\Rightarrow$

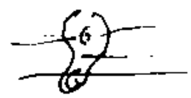
~~$\phi_p(\mathbb{Q})$~~ i.e. $\Longrightarrow$
~~~~

Then $\phi_p(v)$ is either 0 for all p, or

$\prod_{p_i = \text{places of } \mathbb{Q}} \phi_{p_i}(v) = 1$.   [Note: we've fixed global additive character, $\prod \psi_{p_i} = 1$]

[Why? $\phi_p(a_i, b_i) = \psi_p(\sum a_i b_i) = 1$]

quadratic form ~~skew~~

Picture: arithmetic curve $\begin{array}{c} C \\ \downarrow \pi \\ \overline{\operatorname{Spec}\mathbb{Z}} \end{array}$, 

for each $v \in \overline{\operatorname{Spec}\mathbb{Z}}$ have conformal field theory on p-adic curve = neighborhood of fiber of $\pi^{-1}(v)$

CFT here: $V = D(F((t)))$

For every $x \in C_v$, $D(F((t)))$ is an automorphic representation of Heisenberg X wrt subgroup

$$\left(\Gamma(C \setminus x, \mathcal{O}) \times \Gamma(C \setminus x, \omega) \times F\right) \rtimes SL_2\begin{pmatrix} \mathcal{O} & \omega \\ \omega^{-1} & \mathcal{O} \end{pmatrix}$$

Oscillator rep

For a horizontal divisor $\begin{array}{c} C \\ \downarrow \pi \uparrow s \\ \overline{\operatorname{Spec}\mathbb{Z}} \end{array}$ $\overline{C_s}$ = set of places of $s(\operatorname{Spec}\mathbb{Z})$ … i.e. residue field is a number field

Q: What are automorphic forms on such horizontal divisors?

Want automorphic functional ϕ on $\mathbb{R}((t)) \times \mathbb{R}((t))\,dt \times \mathbb{R} \rtimes \widetilde{SL_2}\,\mathbb{R}((t))$

wrt $\Gamma = \mathbb{Z}((t)) \times \mathbb{Z}((t))\,dt \rtimes \widetilde{SL_2}\,\mathbb{Z}((t))$