

# Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation

Luis Caffarelli\*, Alexis Vasseur†

Department of Mathematics  
University of Texas at Austin

**Abstract:** Motivated by the critical dissipative quasi-geostrophic equation, we prove that drift-diffusion equations with  $L^2$  initial data and minimal assumptions on the drift are locally Hölder continuous. As an application we show that solutions of the quasi-geostrophic equation with initial  $L^2$  data and critical diffusion  $(-\Delta)^{1/2}$ , are locally smooth for any space dimension.

## 1 Introduction

Non linear evolution equations with fractional diffusion arise in many contexts: In the quasi-geostrophic flow model (Constantin [4]), in boundary control problems (Duvaut-Lions [9]), in surface flame propagation and in financial mathematics. In this paper, motivated by the quasi-geostrophic model, we study the equation:

$$\begin{aligned}\partial_t \theta + v \cdot \nabla \theta &= -\Lambda \theta, & x \in \mathbb{R}^N, \\ \operatorname{div} v &= 0,\end{aligned}\tag{1}$$

where  $\Lambda \theta = (-\Delta)^{1/2} \theta$ . The main two theorems are roughly the following a priori estimates:

**Theorem 1** (From  $L^2$  to  $L^\infty$ ). *Let  $\theta(t, x)$  be a function in  $L^\infty(0, T; L^2(\mathbb{R}^N)) \cap L^2(0, T; H^{1/2}(\mathbb{R}^N))$ . For every  $\lambda > 0$  we define:*

$$\theta_\lambda = (\theta - \lambda)_+.$$

*If  $\theta$  (and  $-\theta$ ) satisfies for every  $\lambda > 0$  the level set energy inequalities:*

$$\begin{aligned}\int_{\mathbb{R}^N} \theta_\lambda^2(t_2, x) dx + 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |\Lambda^{1/2} \theta_\lambda|^2 dx dt \\ \leq \int_{\mathbb{R}^N} \theta_\lambda^2(t_1, x) dx, & \quad 0 < t_1 < t_2,\end{aligned}$$

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\*caffarel@math.utexas.edu

†vasseur@math.utexas.edu

then:

$$\sup_{x \in \mathbb{R}^N} |\theta(T, x)| \leq C^* \frac{\|\theta_0\|_{L^2}}{T^{N/2}}.$$

**Remark:** That solutions to equation (1) are expected to satisfy the energy inequality follows from writing  $\Lambda$  as the normal derivative of the harmonic extension of  $\theta$  to the upper half space. Existence theory is sketched in appendix C. In the case of the Quasi-geostrophic equation it can also be seen as a corollary of Cordoba and Cordoba [7].

Those energy inequalities are reminiscent of the notion of entropic solutions for scalar conservation laws. Consider a weak solution of (1) lying in  $L^2(H^{1/2})$  and for which we can define the equality (in the sense of distribution for example):

$$\phi'(\theta)v \cdot \nabla \theta = \operatorname{div}(v\phi(\theta)),$$

for any Lipschitz function  $\phi$ . Then  $\theta$  verifies the level set energy inequalities. In the case of the Quasi-geostrophic equation,  $v \in L^2(H^{1/2})$  and we can give a sense to:

$$v \cdot \nabla \phi(\theta).$$

Indeed, using the harmonic extension, it can be shown that if  $\theta$  lies in  $L^2(H^{1/2})$  so does  $\phi(\theta)$ . and so  $\nabla \phi(\theta)$  lies in  $L^2(H^{-1/2})$ .

For the second theorem, (from  $L^\infty$  to  $C^\alpha$ ), we need better control of  $v$ :

**Theorem 2** (From  $L^\infty$  to  $C^\alpha$ ). We define  $Q_r = [-r, 0] \times [-r, r]^N$ , for  $r > 0$ . Assume now that  $\theta(t, x)$  is bounded in  $[-1, 0] \times \mathbb{R}^N$  and  $v|_{Q_1} \in L^\infty(-1, 0; BMO)$ , then  $\theta$  is  $C^\alpha$  in  $Q_{1/2}$ .

**Remark1:** The global bound of  $\theta$  is not really necessary, only local  $L^\infty$  and integrability at infinity against the Poisson kernel, as we will see later.

**Remark2:** Note that both theorems depend only on the resulting energy inequality and not on the special form of  $\Lambda$ .

From these two theorems, the regularity of solutions to the quasi-geostrophic equation follows.

**Theorem 3** Let  $\theta$  be a solution to an equation

$$\begin{aligned} \partial_t \theta + u \cdot \nabla \theta &= -\Lambda \theta, & x \in \mathbb{R}^N, \\ \operatorname{div} u &= 0, \end{aligned} \tag{2}$$

with

$$u_j = \bar{R}_j[\theta], \tag{3}$$

$\bar{R}_j$  a singular integral operator. Assume also that  $\theta$  verifies the level set energy inequalities stated in Theorem 1. Then, for every  $t_0 > 0$  there exists  $\alpha$  such that  $\theta$  is bounded in  $C^\alpha([t_0, \infty[ \times \mathbb{R}^N)$ .

Indeed, Theorem 1 gives that  $\theta$  is uniformly bounded on  $[t_0, \infty[$  for every  $t_0 > 0$ . Singular integral operators are bounded from  $L^\infty$  to  $BMO$ . This gives that  $u \in L^\infty(t_0, \infty; BMO(\mathbb{R}^N))$  and, after proper scaling, Theorem 2 gives the result of Theorem 3.

**Remark 1:** Higher regularity then follows from standard potential theory, by noticing that the fundamental solution of the operator:

$$\partial_t + \Lambda\theta = 0$$

is the Poisson kernel and that in the non linear term we can subtract a constant both  $\theta_0$  from  $\theta$  and  $u_0$  from  $u$ , this last one by a change of coordinates:

$$x^* = x - tu_0,$$

doubling its Holder decay (see appendix).

Strictly speaking, the dissipative quasi-geostrophic flow model in the critical case corresponds to the case  $N = 2$  and

$$\begin{aligned} u_1 &= -R_2\theta, \\ u_2 &= R_1\theta, \end{aligned}$$

where  $R_i$  is the usual Riesz transform defined from the Fourier transform:  $\widehat{R_i\theta} = \frac{i\xi_i}{|\xi|}\widehat{\theta}$ . This model was introduced by some authors as a toy model to investigate the global regularity of solutions to 3D fluid mechanics (see for instance [4]). When replacing the diffusion term  $-\Lambda$  by  $-\Lambda^\beta$ ,  $0 \leq \beta \leq 2$ , the situation is classically decomposed into 3 cases according to the order of diffusion versus transport: The subcritical case for  $\beta > 1$ , the critical case for  $\beta = 1$  and the supercritical case for  $\beta < 1$ .

Weak solutions has been constructed by Resnick in [12]. Constantin and Wu showed in [6] that in the subcritical case any solution with smooth initial value is smooth for all time. Constantin Cordoba and Wu showed in [5] that the regularity is conserved for all time in the critical case provided that the initial value is small in  $L^\infty$ . In both the critical case and supercritical cases, Chae and Lee considered in [3] the well-posedness of solutions with initial conditions small in Besov spaces (see also Wu [16]).

Notice that our case corresponds to the critical case and global regularity in  $C^{1,\beta}$ ,  $\beta < 1$  is showed for any initial value in the energy space without hypothesis of smallness. This ensures that the solutions are classical.

Let us also cite a result of maximum principle due to Cordoba and Cordoba [7], results of behavior in large time due to Schonbeck and Schonbeck [14], [13], and a criteria for blow-up in Chae [2].

**Remark 2:** In a recently posted preprint in arXiv, Kiselev, Nazarov, and Volberg present a very elegant proof of the fact that in 2D, solutions with periodic  $C^\infty$  data for the quasi-geostrophic equation remain  $C^\infty$  for all time ([10]).

We conclude our introduction by pointing out that our techniques also can be seen as a parabolic De Giorgi Nash Moser method to treat "boundary parabolic problems" of the type:

$$\begin{aligned} \operatorname{div}(a\nabla\theta) &= 0, & \text{in } \Omega \times [0, T] \\ [f(\theta)]_t &= \theta_\nu & \text{on } \partial\Omega \times [0, T], \end{aligned}$$

that arise in boundary control (see Duvaut Lions [9]). Note also that similar results to Theorem 1 can be obtained even for systems (See Vasseur [15] and Mellet, Vasseur [11] for applications of the method in fluid mechanics).

## 2 $L^\infty$ bounds

This section is devoted to the proof of Theorem 1. The simple proof is based on a recurrence non linear relation between consecutive truncations of  $\theta$  at an increasing sequence of levels. Following the ideas of De Giorgi, this is attained thanks to the interplay between the energy inequality that controls  $|\nabla\theta|$  by  $\theta$ , and the opposite effect of the Sobolev inequality that controls  $\theta$  by  $\nabla\theta$ , and the different homogeneity of these inequalities.

We use the truncation energy inequality for the levels:

$$\lambda = C_k = M(1 - 2^{-k}),$$

where  $M$  will be chosen later. This leads to the following energy inequality for the truncation function  $\theta_k = (\theta - C_k)_+$ :

$$\partial_t \int_{\mathbb{R}^N} \theta_k^2 dx + 2 \int_{\mathbb{R}^N} |\Lambda^{1/2} \theta_k|^2 dx \leq 0. \quad (4)$$

Let us fix a  $t_0 > 0$ , we want to show that  $\theta$  is bounded for  $t > t_0$ . We introduce  $T_k = t_0(1 - 2^{-k})$ , and the level set of energy/dissipation of energy:

$$U_k = \sup_{t \geq T_k} \left( \int_{\mathbb{R}^N} \theta_k^2 dx \right) + 2 \int_{T_k}^{\infty} \int_{\mathbb{R}^N} |\Lambda^{1/2} \theta_k|^2 dx dt.$$

integrating (4) in time between  $s$ ,  $T_{k-1} < s < T_k$ , and  $t > T_k$  and between  $s$  and  $+\infty$  we find:

$$U_k \leq 2 \int_{\mathbb{R}^N} \theta_k^2(s) dx.$$

Taking the mean value in  $s$  on  $[T_{k-1}, T_k]$  we find:

$$U_k \leq \frac{2^{k+1}}{t_0} \int_{T_{k-1}}^{\infty} \int_{\mathbb{R}^N} \theta_k^2 dx dt. \quad (5)$$

We want to control the right-hand side by  $U_{k-1}$  in a non linear way. Sobolev and Holder inequalities give:

$$U_{k-1} \geq C \|\theta_{k-1}\|_{L^{\frac{2(N+1)}{N}}(]T_{k-1}, \infty[ \times \mathbb{R}^N)}^2.$$

Note that if  $\theta_k > 0$  then  $\theta_{k-1} \geq 2^{-k}M$ . So

$$\mathbf{1}_{\{\theta_k > 0\}} \leq \left( \frac{2^k}{M} \theta_{k-1} \right)^{2/N}.$$

Hence:

$$\begin{aligned} U_k &\leq \frac{2^{k+1}}{t_0} \int_{T_{k-1}}^{\infty} \int_{\mathbb{R}^N} \theta_{k-1}^2 \mathbf{1}_{\{\theta_k > 0\}} dx dt \\ &\leq 2 \frac{2^{\frac{N+2}{N}k}}{t_0 M^{2/N}} \int_{T_{k-1}}^{\infty} \int_{\mathbb{R}^N} \theta_{k-1}^{2\frac{N+1}{N}} dx dt \\ &\leq 2C \frac{2^{\frac{N+2}{N}k}}{t_0 M^{2/N}} U_{k-1}^{\frac{N+1}{N}}. \end{aligned}$$

For  $M$  such that  $M/t_0^{N/2}$  is big enough (depending on  $U_0$ ) we have  $U_k$  which converges to 0. This gives  $\theta \leq M$  for  $t \geq t_0$ . The same proof on  $-\theta$  gives the same bound for  $|\theta|$ . Note that  $U_0 \leq \|\theta_0\|_{L^2}^2$ . The scaling invariance  $\theta_\varepsilon(s, y) = \theta(\varepsilon s, \varepsilon y)$  gives the final dependence with respect to  $\|\theta_0\|_{L^2}$ .  $\square$

This theorem leads to the following corollary.

**Corollary 4** *There exists a constant  $C^* > 0$  such that any solution  $\theta$  of (2) (3) verifies:*

$$\begin{aligned} \sup_{x \in \mathbb{R}^N} |\theta(T, x)| &\leq C^* \frac{\|\theta_0\|_{L^2(\mathbb{R}^N)}}{T^{N/2}}, \\ \|u(T, \cdot)\|_{BMO(\mathbb{R}^N)} &\leq C^* \frac{\|\theta_0\|_{L^2(\mathbb{R}^N)}}{T^{N/2}}. \end{aligned}$$

**Proof.** First note that the property on  $u$  follows directly from the property on  $\theta$  and the imbedding of the Riesz function from  $L^\infty$  to  $BMO$ . We make use of the following result of Cordoba and Cordoba (see [7]): for any convex function  $\phi$  we have the pointwise inequality:

$$-\phi'(\theta)\Lambda\theta \leq -\Lambda(\phi(\theta)).$$

Making use of this inequality with:

$$\phi_k(\theta) = (\theta - C_k)_+ = \theta_k$$

leads to:

$$\partial_t \theta_k + u \cdot \nabla \theta_k \leq -\Lambda \theta_k.$$

Multiplying by  $\theta_k$  and integrating in  $x$  gives (4), using that  $u$  is divergence free.  $\square$

**Remark:** We point out that the level set energy inequalities we assume in Theorem 1 is heuristically a general fact (See appendix C).

### 3 Local energy inequality:

In order to develop the Holder regularity method, it is necessary to localize by space and time truncation the energy inequality above. Due to the non-locality of the diffusion operator, this appears complicated. Fortunately,  $\Lambda\theta$  can be thought as the normal derivative of a harmonic extension of  $\theta$  (the Dirichlet to Neuman operator of  $\theta$ ). This allows us to realize the truncation as a standard local one in one more dimension: We introduce first the harmonic extension  $L$  defined from  $C_0^\infty(\mathbb{R}^N)$  to  $C_0^\infty(\mathbb{R}^N \times \mathbb{R}^+)$  by:

$$\begin{aligned} -\Delta L(\theta) &= 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ L(\theta)(x, 0) &= \theta(x) & \text{for } x \in \mathbb{R}^N \end{aligned}$$

(This extension consists simply in convolving  $\theta$  with the Poisson kernel of the upper half space in one more variable. See [1] for a general discussion). Then the following result holds true: consider  $\theta$  defined on  $\mathbb{R}^N$ . then:

$$\Lambda\theta(x) = \partial_\nu[L\theta](x), \tag{6}$$

where we denote  $\partial_\nu[L\theta]$  the normal derivative of  $L\theta$  on the boundary  $\{(x, 0) | x \in \mathbb{R}^N\}$ .

In the following, we will denote the harmonic extension of  $\theta$  by:

$$\theta^*(t, x, z) = L(\theta(t, \cdot))(x, z). \tag{7}$$

We denote  $B_r = [-r, r]^N$  a cube in the  $x$  variable only,  $B_r^* = B_r \times (0, r) \in \mathbb{R}^N \times (0, \infty)$  a cube in the  $x, z$  variables, sitting on the  $z = 0$  plane, and  $[y]_+ = \sup(0, y)$ .

The rest of this section is devoted to the proof of the following proposition, a local energy inequality in the  $x, z$  variables. The effect of the nonlocal part of  $\Lambda$  becomes encoded locally in the extra variable. At this point, we know already that for any positive time  $T > 0$ :

$$\|\theta\|_{L^2(\mathbb{R}^N)} + \|\theta\|_{L^\infty(\mathbb{R}^N)} \leq C$$

uniformly in  $t > T$ , and for the application we have in mind, the quasi geostrophic equation, this implies that

$$\|v\|_{L^2(\mathbb{R}^N)} + \|v\|_{BMO(\mathbb{R}^N)} \leq C,$$

uniformly in  $t > T$ . This is, therefore, the main hypothesis below.

**Proposition 5** *Let  $t_1, t_2$  be such that  $t_1 < t_2$  and let  $\theta \in L^\infty(t_1, t_2; L^2(\mathbb{R}^N))$  with  $\Lambda^{1/2}\theta \in L^2((t_1, t_2) \times \mathbb{R}^N)$ , be solution to (1) with a velocity  $v$  satisfying:*

$$\|v\|_{L^\infty(t_1, t_2; BMO(\mathbb{R}^N))} + \sup_{t_1 \leq t \leq t_2} \left| \int_{B_2} v(t, x) dx \right| \leq C_v. \tag{8}$$

Then there exists a constant  $\Phi$  (depending only on  $C_v$ ) such that for every  $t_1 \leq t \leq t_2$  and cut-off function  $\eta$  such that the restriction of  $\eta[\theta^*]_+$  on  $B_2^*$  is compactly supported in  $B_2 \times (-2, 2)$ :

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{B_2^*} |\nabla(\eta[\theta^*]_+)|^2 dx dz dt + \int_{B_2} (\eta[\theta]_+)^2(t_2, x) dx \\ & \leq \int_{B_2} (\eta[\theta]_+)^2(t_1, x) dx + \Phi \int_{t_1}^{t_2} \int_{B_2} (|\nabla\eta|[\theta]_+)^2 dx dt \\ & \quad + 2 \int_{t_1}^{t_2} \int_{B_2^*} (|\nabla\eta|[\theta^*]_+)^2 dx dz dt. \end{aligned} \quad (9)$$

**Remark:** Note that, as a difference with the standard parabolic estimates, this energy inequality controls  $\|\eta\theta\|_{L_t^\infty(L_x^2)}$  and  $\|\nabla(\eta\theta^*)\|_{L_t^2(L_{xz}^2)}$ . We are missing, in some sense,  $\|\eta\theta^*\|_{L_t^p(L_{xz}^2)}$  that would provide the link between  $\nabla(\eta\theta^*)$  and  $\eta\theta$ . In order to control  $\eta\theta^*$  we will have to make a careful decomposition of  $\eta\theta^*$  as the part coming from  $\eta\theta$  as boundary value, and the rest coming from "far away" (Step 5: Propagation of the support property (12) and the proof of Lemma 6).

**Proof.** We have for every  $t_1 < t < t_2$ :

$$\begin{aligned} 0 &= \int_{B_2^*} \eta^2[\theta^*]_+ \Delta\theta^* dx dz \\ &= - \int_{B_2^*} |\nabla(\eta[\theta^*]_+)|^2 dx dz + \int_{B_2^*} |\nabla\eta|^2 [\theta^*]_+^2 dx dz \\ & \quad + \int_{B_2} \eta^2[\theta]_+ \Lambda\theta dx. \end{aligned}$$

Using equation (1), we find that:

$$\begin{aligned} & - \int_{B_2} \eta^2[\theta]_+ \Lambda\theta dx \\ &= \frac{\partial}{\partial t} \left( \int_{B_2} \eta^2 \frac{[\theta]_+^2}{2} dx \right) - \int_{B_2} \nabla\eta^2 \cdot v \frac{[\theta]_+^2}{2} dx. \end{aligned}$$

This leads to:

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{B_2^*} |\nabla(\eta[\theta^*]_+)|^2 dx dz ds + \int_{B_2} \eta^2 \frac{[\theta]_+^2(t_2)}{2} dx \\ & \leq \int_{B_2} \eta^2 \frac{[\theta]_+^2(t_1)}{2} dx + \int_{t_1}^{t_2} \int_{B_2^*} |\nabla\eta|^2 [\theta^*]_+^2 dx dz ds \\ & \quad + \left| \int_{t_1}^{t_2} \int_{B_2} \eta \nabla\eta \cdot v [\theta]_+^2 dx ds \right|. \end{aligned}$$

To dominate the last term, we first use the Trace theorem and Sobolev imbed-

ding to find:

$$\begin{aligned}
\|\mathbf{1}_{\{B_2\}}\eta\theta_+\|_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)}^2 &\leq C \|\mathbf{1}_{\{B_2\}}\eta\theta_+\|_{H^{1/2}(\mathbb{R}^N)}^2 = C \int_{B_2} (\eta\theta_+) \Delta(\mathbf{1}_{\{B_2\}}\eta\theta_+) dx \\
&= C \int_0^\infty \int_{\mathbb{R}^N} |\nabla L(\mathbf{1}_{\{B_2\}}\eta\theta_+)|^2 dx dz \\
&\leq C \int_0^\infty \int_{\mathbb{R}^N} |\nabla[\mathbf{1}_{\{B_2^*\}}\eta(\theta^*)_+]|^2 dx dz \\
&= C \int_{B_2^*} |\nabla[\eta(\theta^*)_+]|^2 dx dz.
\end{aligned}$$

In the last inequality, we have used the support property of  $\eta(\theta^*)_+$ . In the second last inequality we have used the fact that  $L(\mathbf{1}_{\{B_2\}}\eta\theta_+)$  is harmonic and have the same trace as  $\mathbf{1}_{\{B_2^*\}}\eta(\theta^*)_+$  at  $z = 0$ . Therefore we split:

$$\begin{aligned}
&\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \nabla\eta^2 \cdot v \frac{[\theta]_+^2}{2} dx ds \right| \\
&\leq \varepsilon \int_{t_1}^{t_2} \|\eta\theta_+\|_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)}^2 ds + \frac{1}{\varepsilon} \int_{t_1}^{t_2} \|[\nabla\eta]v[\theta]_+\|_{L^{\frac{2N}{N+1}}(\mathbb{R}^N)}^2 ds.
\end{aligned}$$

The first term is absorbed by the left. The second can be bounded, using Holder inequality, by:

$$\frac{1}{\varepsilon} \|v\|_{L^\infty(t_1, t_2; L^{2N}(B_2))}^2 \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |[\nabla\eta][\theta]_+|^2 dx ds,$$

which gives the desired result.  $\square$

## 4 From $L^2$ to $L^\infty$ :

In these two sections (4 and 5) we follow De Giorgi's ideas in his classical proof of the Holder continuity of solutions to elliptic equations (see [8]). The first step is a local, scalable version of the  $L^\infty$  bound above. It establishes that the space time localization of a level truncation of  $\theta$  is bounded by the  $L^2$  norm. The second step is the so called "oscillation lemma". we give a rough description of this lemma. Suppose that  $\theta$  oscillates in  $Q_1 = [-1, 0] \times B_1$  between  $-2$  and  $2$ , but it is negative most of the time. In particular, if  $\|\theta_+\|_{L^2}$  is very small, then the local  $L^2$  to  $L^\infty$  bound mentioned above, will imply that (in a small domain)  $\theta^+$  is very small. In particular, we prove that  $\theta_+|_{Q_{1/2}=[-1/2, 0] \times B_{1/2}} \leq 2 - \lambda$ , effectively reducing the oscillations of  $\theta$  by  $\lambda$  (see Lemma 6). Of course, we do not know a priori that  $\|\theta_+\|_{L^2}$  is very small.

But we only know that in  $Q_1$ ,  $\theta$  is at least half of the time positive, or negative, say negative. We then have to reproduce a version of De Giorgi's isoperimetric inequality that says that to go from zero to one  $\theta$  needs "some room" (section 5). Therefore the set  $\{\theta \leq 1\}$  is "strictly larger" than the set  $\{\theta \leq 0\}$

(see Lemma 8). Repeating this argument at truncation levels  $C_k = 2 - 2^{-k}$ , we fall, after a finite number of steps,  $k_0$ , into the first case, effectively diminishing the oscillations of  $\theta$  by  $\lambda 2^{-k_0}$ . This implies Holder continuity (section 6 and section 7).

This section is devoted to the first step of the proof, the  $L^2$  to  $L^\infty$  lemma. It says that, under suitable conditions on  $v$ , we can control the  $L^\infty$  norm of  $\theta$  from the  $L^2$  norm of both  $\theta$  and  $\theta^*$  locally.

**Lemma 6** *We assume, as in section 3, that*

$$\|v\|_{L^\infty(-4,0;BMO(\mathbb{R}^N))} + \sup_{-4 \leq t \leq 0} \left| \int_{B_4} v(t,x) dx \right| \leq C_v.$$

*Then, there exists  $\varepsilon_0 > 0$  (depending only on  $N$  and  $C_v$ ), and  $\lambda > 0$  (depending only on  $N$ ) such that for every  $\theta$  solution to (1) the following property holds true.*

*If we have:*

$$\theta^* \leq 2 \quad \text{in } [-4, 0] \times B_4^*,$$

*and*

$$\int_{-4}^0 \int_{B_4^*} (\theta^*)_+^2 dx dz ds + \int_{-4}^0 \int_{B_4} (\theta)_+^2 dx ds \leq \varepsilon_0,$$

*then:*

$$(\theta)_+ \leq 2 - \lambda \quad \text{on } [-1, 0] \times B_1.$$

**Remark:** Note that this is not a "pure"  $L^2$  to  $L^\infty$  estimate, since we assume that  $\theta^*$  is already bounded by 2. Nevertheless, since our final objective is an improvement of the oscillation of  $\theta^*$ , the gain from 2 to  $2 - \lambda$  will suffice.

**Proof.** We split the proof of the lemma into several steps. Steps 1 and 2 are preliminary views. We construct auxiliary barriers and recurrence constants. Step 3 describes which is the recurrence relation we are aiming for. The actual proof really starts in step 4. As in the proof of theorem 1 at the beginning of the paper, we will now consider a sequence of truncations for an increasing sequence of levels  $C_k$  converging to  $2 - \lambda$ , and will prove that by the time we reach  $2 - \lambda$ , the corresponding truncation has zero energy and is, thus, identically zero. Before giving into the proof, we give an informal description of the arguments involved. In principle, we would like to prove that if  $\theta_+^*$  is less than 2 in the cylinder  $B_4^* \times [-4, 0]$  and both  $\theta_+^*$  and  $\theta_+$  have very small  $L^2$  norm, then they are both less than  $2 - \lambda$  in  $B_1^* \times [-1, 0]$ . Thanks to the barrier  $b_1$  below, we notice that it is enough that  $\theta_+$  be below  $2 - \lambda$ , since  $\theta_+^*$  is bounded by  $b_1$  plus the harmonic extension of  $\theta_+$ . The second observation we make is that if the  $L^2$  norm of  $\theta_+$  is small,  $\theta_+^*$  dips to very small values for  $z$  small, (proportionally in some way to  $\|\theta_+\|_{L^2}$ ). Indeed, the part corresponding to  $b_1$  goes linearly to zero while the Poisson kernel smoothes  $\theta_+$ . In other words the influence of the global part of  $\Lambda$ , reflected by  $b_1$ , decays linearly as  $z$  goes to zero, and we can almost eliminate as we truncate  $\theta$  at increasing levels (Step 5, propagation of

the support property). This eliminates the need of a truncation in  $z$  and almost eliminates the influence of the global part of  $\Lambda$ . It only remains on the small "lateral edges", that is where the space truncation takes place and for very small  $z$ , and that has exponential decay (barrier  $b_2$ ). That is what allows us to obtain the appropriate recurrence relation for  $A_k$ , that implies that " $\theta_\infty = 0$ " on  $B_1 \times [-1, 0]$ , i.e.  $\theta \leq 2 - \lambda$ .

Step 4 is a first "long jump" to  $k \geq 12N$ , that puts us into the appropriate configuration described above, to start the inductive process provided  $\theta_+$  is small enough.

**Step1. Useful barrier functions:** The following two barrier functions,  $b_1$  and  $b_2$ , will be used to control how the values of  $\theta^*$  far from the "disc"  $D_1^* = B_1 \times \{0\}$  influence  $\theta^*$  near  $D_1^*$ .

Consider the function  $b_1$ , defined by:

$$\begin{aligned} \Delta b_1 &= 0 && \text{in } B_4^* \\ b_1 &= 2 && \text{on the sides of the cube } B_4^* \text{ except for } z = 0 \\ b_1 &= 0 && \text{for } z = 0. \end{aligned}$$

Then there exists  $\lambda > 0$  such that:

$$b_1(x, z) \leq 2 - 4\lambda \text{ on } B_2^*.$$

This result follows directly from the maximum principle.

We consider now  $b_2$  harmonic function defined by:

$$\begin{aligned} \Delta b_2 &= 0 && \text{in } [0, \infty[ \times [0, 1], \\ b_2(0, z) &= 2 && 0 \leq z \leq 1, \\ b_2(x, 0) &= b_2(x, 1) = 0 && 0 < x < \infty. \end{aligned}$$

Then there exists  $\bar{C} > 0$  such that:

$$|b_2(x, z)| \leq \bar{C}e^{-x/2}. \quad (10)$$

Notice that  $\bar{C}$  is universal. Indeed we can see easily that

$$b_2(x, z) \leq 2\sqrt{2} \cos(z/2)e^{-x/2},$$

since this function is harmonic and bigger than  $b_2$  on the boundary.

**Step 2. Setting of constants:** In this step we fix a set of constants. We make the choice to set them right away to convince the reader that the proof is not circular.

**Lemma 7** *There exist  $0 < \delta < 1$  and  $M > 1$  such that for every  $k > 0$ :*

$$\begin{aligned} 2N\bar{C}e^{-\frac{2-k}{2\delta^k}} &\leq \lambda 2^{-k-2}, \\ \frac{M^{-k/2}}{\delta^{k+1}} \|P(1)\|_{L^2} &\leq \lambda 2^{-k-2}, \\ M^{-k} &\geq C_0^k M^{-(1+1/N)(k-3)} \quad k \geq 12N, \end{aligned}$$

where  $\bar{C}$  is defined from step 1,  $P(1)$  is the value at  $z = 1$  of the Poisson kernel  $P(z)(x)$ , and  $C_0$  is defined by (16).

The proof is easy. We construct first  $\delta$  to verify the first inequality in the following way. If  $\delta < 1/4$ , the inequality is true for  $k > k_0$  due to the exponential decay. If necessary, we then choose  $\delta$  smaller to make the inequality also valid for  $k < k_0$ . Now that  $\delta$  has been fixed, we have to choose  $M$  large to satisfy the remaining inequalities. Note that the second inequality is equivalent to:

$$\left(\frac{2}{\delta\sqrt{M}}\right)^k \leq \frac{\lambda\delta}{4\|P(1)\|_{L^2}}.$$

It is so sufficient to take:

$$M \geq \sup\left(\frac{2}{\delta}, \frac{8\|P(1)\|_{L^2}}{\lambda\delta^2}\right)^2.$$

The third inequality is equivalent to:

$$\left(\frac{M}{C_0^N}\right)^{k/N} \geq M^{3(1+1/N)}.$$

For this case it is sufficient to take  $M \geq \sup(1, C_0^{2N})$ . Indeed, this ensures  $M^2/C_0^{2N} \geq M$  and so:

$$\left(\frac{M}{C_0^N}\right)^{k/N} \geq M^{k/(2N)} \geq M^6,$$

for  $k \geq 12N$ . But  $M^6 \geq M^{3(1+1/N)}$  for  $M \geq 1$  and  $N \geq 2$ .

Therefore we can fix:

$$M = \sup\left(1, C_0^{2N}, \left(\frac{2}{\delta}\right)^2, \left(\frac{8\|P(1)\|_{L^2}}{\lambda\delta^2}\right)^2\right).$$

The constant  $\lambda$ ,  $\delta$ , and  $M$  are now fixed for the rest of the proof. The constant  $\varepsilon_0$  will be constructed from those ones.

**Step 3. Induction:** We set:

$$\theta_k = (\theta - C_k)_+, \quad \theta_k^* = (\theta^* - C_k)_+,$$

with  $C_k = 2 - \lambda(1 + 2^{-k})$ . Note that  $\theta_k^* \neq (\theta_k)^*$ . We consider a cut-off function in  $x$  only such that:

$$\begin{aligned} \mathbf{1}_{\{B_{1+2^{-k-1}}\}} &\leq \eta_k \leq \mathbf{1}_{\{B_{1+2^{-k}}\}}, \\ |\nabla\eta_k| &\leq C2^k, \end{aligned}$$

and we denote:

$$A_k = \int_{-1-2^{-k}}^0 \int_0^{\delta^k} \int_{\mathbb{R}^N} |\nabla(\eta_k\theta_k^*)|^2 dx dz dt + \sup_{t \in [-1-2^{-k}, 0]} \int_{\mathbb{R}^N} (\eta_k\theta_k)^2 dx.$$

We want to prove simultaneously that for every  $k \geq 0$ :

$$A_k \leq M^{-k} \quad (11)$$

$$\eta_k \theta_k^* = 0 \quad \text{for } \delta^k \leq z \leq 2. \quad (12)$$

**Step 4. Initial step:** We prove in this step that if  $\varepsilon_0$  is small enough, then (11) is verified for  $0 \leq k \leq 12N$ , and that (12) is verified for  $k = 0$ . We use the energy inequality (9) with cut-off function  $\eta_k(x)\psi(z)$  where  $\psi$  is a fixed cut-off function in  $z$  only. Taking the mean value of (9) in  $t_1$  between  $-4$  and  $-2$ , we find that (11) is verified for  $0 \leq k \leq 12N$  if  $\varepsilon_0$  is taken such that:

$$C2^{24N}(1 + \Phi)\varepsilon_0 \leq M^{-12N}. \quad (13)$$

We have used that  $|\nabla \eta_k|^2 \leq C2^{24N}$  for  $0 \leq k \leq 12N$ . Let us consider now the support property (12). By the maximum principle, we have:

$$\theta^* \leq (\theta_+ \mathbf{1}_{B_4}) * P(z) + b_1(x, z),$$

in  $\mathbb{R}^+ \times B_4^*$ , where  $P(z)$  is the Poisson kernel. Indeed, the right-hand side function is harmonic, positive and the trace on the boundary is bigger than the one of  $\theta^*$ .

From step 1 we have:  $b_1(x, z) \leq 2 - 4\lambda$ . Moreover:

$$\|\theta_+ \mathbf{1}_{B_4} * P(z)\|_{L^\infty(z \geq 1)} \leq C\|P(1)\|_{L^2} \sqrt{\varepsilon_0} \leq C\sqrt{\varepsilon_0}.$$

Choosing  $\varepsilon_0$  small enough such that this constant is smaller than  $2\lambda$  gives:

$$\theta^* \leq 2 - 2\lambda \quad \text{for } 1 \leq z \leq 2, t \geq 0, x \in B,$$

so:

$$\theta_0^* = (\theta^* - (2 - 2\lambda))_+ \leq 0 \quad \text{for } 1 \leq z \leq 2, t \geq 0, x \in B.$$

Hence  $\eta_0 \theta_0^*$  vanishes for  $1 = \delta^0 \leq z \leq 2$ .

**Step 5. Propagation of the support property (12):** Assume that (11) and (12) are verified at  $k$ . We want to show that (12) is verified at  $(k+1)$ . We will show also that the following is verified at  $k$ :

$$\eta_k \theta_{k+1}^* \leq [(\eta_k \theta_k) * P(z)] \eta_k, \quad \text{on } \overline{B}_k^*, \quad (14)$$

where  $\overline{B}_k^* = B_{1+2^{-k}} \times [0, \delta^k]$ . We want to control  $\theta_k^*$  on this set by harmonic functions taking into account the contributions of the sides one by one. On  $z = \delta^k$  we have no contribution thanks to the induction property (12) at  $k$  (the trace is equal to 0). The contribution of the side  $z = 0$  can be controlled by:  $\eta_k \theta_k * P(z)$  (It has the same trace as  $\theta_k$  on  $B_{1+2^{-k-1}}$ ).

On each of the other side we control the contribution by the function of  $x = (x_1, \dots, x_N)$ :

$$b_2((x_i - x^+)/\delta^k, z/\delta^k) + b_2((-x_i + x^-)/\delta^k, z/\delta^k),$$

where  $x^+ = (1 + 2^{-k})$  and  $x^- = -x^+$ . Indeed,  $b_2$  is harmonic, and on the side  $x_i^+$  and  $x_i^-$  it is bigger than 2. Finally, by the maximum principle:

$$\theta_k^* \leq \sum_{i=1}^N [b_2((x_i - x^+)/\delta^k, z/\delta^k) + b_2((-x_i + x^-)/\delta^k, z/\delta^k)] + (\eta_k \theta_k) * P(z).$$

From Step 1, for  $x \in B_{1+2^{-k}}$ :

$$\begin{aligned} & \sum_{i=1}^N [b_2((x_i - x^+)/\delta^k, z/\delta^k) + b_2((-x_i + x^-)/\delta^k, z/\delta^k)] \\ & \leq 2N\bar{C}e^{-\frac{2^{-k}}{2\delta^k}} \\ & \leq \lambda 2^{-k-2}, \end{aligned}$$

(thanks to Step 2). This gives (14) since:

$$\theta_{k+1}^* \leq (\theta_k^* - \lambda 2^{-k-1})_+.$$

More precisely, this gives:

$$\theta_{k+1}^* \leq ((\eta_k \theta_k) * P(z) - \lambda 2^{-k-2})_+.$$

So:

$$\eta_{k+1} \theta_{k+1}^* \leq ((\eta_k \theta_k) * P(z) - \lambda 2^{-k-2})_+.$$

From the second property of Step 2, we find for  $\delta^{k+1} \leq z \leq \delta^k$ :

$$\begin{aligned} |(\eta_k \theta_k) * P(z)| & \leq \sqrt{A_k} \|P(z)\|_{L^2} \\ & \leq \frac{M^{-k/2}}{\delta^{k+1}} \|P(1)\|_{L^2} \leq \lambda 2^{-k-2}. \end{aligned}$$

The last inequality makes use of Step 2. Therefore:

$$\eta_{k+1} \theta_{k+1}^* \leq 0 \quad \text{for } \delta^{k+1} \leq z \leq \delta^k.$$

Note, in particular, that with step 4 this gives that (12) is verified up to  $k = 12N + 1$  and (14) up to  $k = 12N$ .

**Step 6. Propagation of Property (11).** We show in this step that if (12) is true for  $k - 3$  and (11) is true for  $k - 3$ ,  $k - 2$  and  $k - 1$  then (11) is true for  $k$ .

First notice that from Step 5, (12) is true at  $k - 2$ ,  $k - 1$ , and  $k$ . We just need to show that:

$$A_k \leq C_0^k (A_{k-3})^{1+1/N} \quad \text{for } k \geq 12N + 1, \quad (15)$$

with:

$$C_0 = C \frac{2^{1+2/N}}{\lambda^{2/N}}. \quad (16)$$

Indeed, if we do the third inequality of Step 2, this will give us the result.

**Step 7. Proof of (15):** Since  $\eta\theta_+^*$  has the same trace at  $z = 0$  that  $(\eta\theta_+)^*$  and the latter is harmonic we have:

$$\int |\nabla(\eta\theta_+^*)|^2 \geq \int |\nabla(\eta\theta_+)^*|^2 = \int |\Lambda^{1/2}(\eta\theta_+)|^2.$$

Sobolev and Holder inequalities give:

$$A_{k-3} \geq C \|\eta_{k-3}\theta_{k-3}\|_{L^{\frac{2(N+1)}{N}}([-1-2^{-k-3}, 0] \times \mathbb{R}^N)}^2.$$

From (14):

$$\|\eta_{k-3}\theta_{k-2}^*\|_{L^{\frac{2(N+1)}{N}}}^2 \leq \|P(1)\|_{L^1}^2 \|\eta_{k-3}\theta_{k-3}\|_{L^{\frac{2(N+1)}{N}}}^2.$$

So:

$$\begin{aligned} A_{k-3} &\geq C \|\eta_{k-3}\theta_{k-2}^*\|_{L^{\frac{2(N+1)}{N}}}^2 + C \|\eta_{k-3}\theta_{k-3}\|_{L^{\frac{2(N+1)}{N}}}^2 \\ &\geq C \left( \|\eta_{k-2}\theta_{k-1}^*\|_{L^{\frac{2(N+1)}{N}}}^2 + \|\eta_{k-2}\theta_{k-1}\|_{L^{\frac{2(N+1)}{N}}}^2 \right). \end{aligned}$$

Since  $\eta_k$  is a cut-off function in  $x$ , and using (14), we have that the restriction to  $B_4^*$  of  $\eta_k\theta_k^*$  is compactly supported in  $B_k \times [-\delta^k, \delta^k]$  in  $x, z$ . We can then use Proposition 5. Taking the mean value of (9) in  $t_1$  between  $-1 - 2^{-k-1}$  and  $-1 - 2^{-k}$ , we find:

$$A_k \leq C2^k(\Phi + 2) \left( \int \eta_{k-1}^2 \theta_k^2 + \int \eta_{k-1}^2 \theta_k^{*2} \right).$$

If  $\theta_k > 0$  then  $\theta_{k-1} \geq 2^{-k}\lambda$ . So:

$$\mathbf{1}_{\{\theta_k > 0\}} \leq \frac{C2^k}{\lambda} \theta_{k-1},$$

and

$$\mathbf{1}_{\{\eta_{k-1} > 0\}} \mathbf{1}_{\{\theta_k > 0\}} \leq \frac{C2^k}{\lambda} \eta_{k-2} \theta_{k-1}.$$

Therefore:

$$A_k \leq \frac{C2^{k(1+2/N)}}{\lambda^{2/N}} A_{k-3}^{1+1/N}.$$

This gives (15).  $\square$

## 5 The second technical lemma.

At this moment we have proven that if  $\theta, \theta^*$  are less than 2, and their  $L^2$  norm is very small in  $B_4 \times [-4, 0]$  then both  $\theta, \theta^*$  are less than  $2 - \lambda$  in  $B_1 \times [-1, 0]$ .

That is, their oscillation decreased. We need to prove now that it is enough that

$$|\{\theta^* \leq 0\}| \geq \frac{|Q_4|}{2}$$

to imply that  $\theta, \theta^*$  are less than  $2 - \lambda$  in  $B_1 \times [-1, 0]$ , since this is our induction hypothesis. This is based on the fact that, due to the energy inequality, there must be a quantitative decay in measure between the consecutive level sets  $\{\theta \geq 0\}$ ,  $\{\theta \geq 1\}$ ,  $\{\theta \geq 2 - 1/2\}$ ,  $\{\theta \geq 2 - 1/4\}$ , etc... Lemma 8 below measures this quantitative separation between  $\{\theta \geq 0\}$  and  $\{\theta \geq 1\}$ , i.e. if  $\{\theta \geq 0\}$  is small then  $\theta$  is very small.

We set  $Q_r = B_r \times [-r, 0]$  and  $Q_r^* = B_r^* \times [-r, 0]$ .

**Lemma 8** *For every  $\varepsilon_1 > 0$ , there exists a constant  $\delta_1 > 0$  with the following property:*

*For every solution  $\theta$  to (1) with  $v$  verifying (8) and:*

$$\begin{aligned} \theta^* &\leq 2 \quad \text{in } Q_4^*, \\ |\{(x, z, t) \in Q_4^*; \theta^*(x, z, t) \leq 0\}| &\geq \frac{|Q_4^*|}{2}, \end{aligned}$$

*we have the following implication:*

*If*

$$|\{(x, z, t) \in Q_4^*; 0 < \{\theta^*(x, z, t) < 1\}| \leq \delta_1$$

*then:*

$$\int_{Q_1} (\theta - 1)_+^2 dx dt + \int_{Q_1^*} (\theta^* - 1)_+^2 dx dz dt \leq C\sqrt{\varepsilon_1}.$$

*Note that  $\delta_1$  depends only on  $N$  and  $C_v$  in (8).*

**Proof.** Take  $\varepsilon_1 \ll 1$ . From the energy inequality (9), we get:

$$\int_{-4}^0 \int_{B_1^*} |\nabla \theta_+^*|^2 dx dz dt \leq C.$$

Let:

$$K = \frac{4 \int |\nabla \theta_+^*|^2 dx dz dt}{\varepsilon_1}.$$

Then:

$$\left| \left\{ t \mid \int_{B_1^*} |\nabla \theta_+^*|^2(t) dx dz \geq K \right\} \right| \leq \frac{\varepsilon_1}{4}. \quad (17)$$

For all  $t \in \{t \mid \int_{B_1^*} |\nabla \theta_+^*|^2(t) dx dz \leq K\}$ , the De Giorgi lemma (see appendix) gives that:

$$|\mathcal{A}(t)| |\mathcal{B}(t)| \leq |\mathcal{C}(t)|^{1/2} K^{1/2},$$

where:

$$\begin{aligned}\mathcal{A}(t) &= \{(x, z) \in B_1^* \mid \theta^*(t, x, z) \leq 0\} \\ \mathcal{B}(t) &= \{(x, z) \in B_1^* \mid \theta^*(t, x, z) \geq 1\} \\ \mathcal{C}(t) &= \{(x, z) \in B_1^* \mid 0 < \theta^*(t, x, z) < 1\}.\end{aligned}$$

Let us set

$$\begin{aligned}\delta_1 &= \varepsilon_1^8, \\ I &= \{t \in [-4, 0]; |\mathcal{C}(t)|^{1/2} \leq \varepsilon_1^3 \text{ and } \int_{B_1^*} |\nabla \theta_+^*|^2(t) dx dz \leq K\}.\end{aligned}$$

First we have, using Tchebichev inequality:

$$\begin{aligned}& \left| \{t \in [-4, 0]; |\mathcal{C}(t)|^{1/2} \geq \varepsilon_1^3\} \right| \\ & \leq \frac{|\{(t, x, z) \mid 0 < \theta^* < 1\}|}{\varepsilon_1^6} \\ & \leq \frac{\delta_1}{\varepsilon_1^6} \\ & \leq \varepsilon_1^2 \leq \varepsilon_1/4.\end{aligned}$$

Hence  $[-4, 0] \setminus I \leq \varepsilon_1/2$ . Secondly we get for every  $t \in I$  such that  $|\mathcal{A}(t)| \geq 1/4$ :

$$|\mathcal{B}(t)| \leq \frac{|\mathcal{C}(t)|^{1/2} K^{1/2}}{|\mathcal{A}(t)|} \leq 4\varepsilon_1^{5/2} \leq \varepsilon_1^2. \quad (18)$$

In particular:

$$\begin{aligned}\int \theta_+^{*2}(t) dx dz &\leq 4(|\mathcal{B}(t)| + |\mathcal{C}(t)|) \\ &\leq 8\varepsilon_1^2.\end{aligned}$$

And so:

$$\int \theta_+^2(t) dx \leq \sqrt{K} \sqrt{\int \theta_+^{*2}(t) dx dz} \leq C\sqrt{\varepsilon_1}.$$

We want to show that  $|\mathcal{A}(t)| > 1/4$  for every  $t \in I \cap [-1, 0]$ . First, since  $|\{(t, x, z) \mid \theta^* \leq 0\}| \geq |Q_4^*|/2$ , there exists  $t_0 \leq -1$  such that  $|\mathcal{A}(t_0)| \geq 1/4$ . So for this  $t_0$ ,  $\int \theta_+^2(t_0) dx \leq C\sqrt{\varepsilon_1}$ . Using the energy inequality (9), we have for every  $t \geq t_0$ :

$$\int \theta_+^2(t) dx \leq \int \theta_+^2(t_0) dx + C(t - t_0).$$

So for  $t - t_0 \leq \delta^* = 1/(64C)$  we have:

$$\int \theta_+^2(t) dx \leq \frac{1}{64}.$$

(Note that  $\delta^*$  do not depend on  $\varepsilon_1$ . Hence we can suppose  $\varepsilon_1 \ll \delta^*$ .) We have:

$$\begin{aligned}\theta_+^*(z) &\leq \theta_+ + \int_0^z \partial_z \theta_+^* dz \\ &\leq \theta_+ + \sqrt{z} \left( \int |\partial_z \theta_+^*|^2 dz \right)^{1/2}.\end{aligned}$$

So, for  $t - t_0 \leq \delta^*$ ,  $t \in I$  and  $z \leq \varepsilon_1^2$  we have:

$$\theta_+^*(t, x, z) \leq \theta_+(t, x) + \left( \varepsilon_1^2 \int |\partial_z \theta_+^*|^2 dz \right)^{1/2}.$$

The integral in  $x$  of the right hand side term is less than  $1/8 + \sqrt{\varepsilon_1} \leq 1/4$ . So by Tchebichev:

$$|\{z \leq \varepsilon_1^2, x \in B_1, \theta_+^*(t) \geq 1\}| \leq \frac{\varepsilon_1^2}{4}.$$

First we work in  $B_1 \times [0, \varepsilon_1^2]$ . Since  $|\mathcal{C}(t)| \leq \varepsilon_1^6$ , this gives

$$\begin{aligned}|\mathcal{A}(t)| &\geq |B_1| \varepsilon_1^2 - |\{z \leq \varepsilon_1^2, x \in B_1, \theta_+^*(t) \geq 1\}| - |\mathcal{C}(t)| \\ &\geq \varepsilon_1^2(1 - 1/4) - \varepsilon_1^6 \geq \varepsilon_1^2/2.\end{aligned}$$

In the same way as (18) we find:

$$|\mathcal{B}(t)| \leq \frac{|\mathcal{C}(t)|^{1/2} K^{1/2}}{\mathcal{A}(t)} \leq 2\sqrt{\varepsilon_1},$$

and:

$$|\mathcal{A}(t)| \geq 1 - 2\sqrt{\varepsilon_1} - \varepsilon_1^6 \geq 1/4.$$

Hence, for every  $t \in [t_0, t_0 + \delta^*] \cap I$  we have:  $|\mathcal{A}(t)| \geq 1/4$ . On  $[t_0 + \delta^*/2, t_0 + \delta^*]$  there exists  $t_1 \in I$  ( $\delta^* \geq \varepsilon_1/4$ ). And so, we can construct an increasing sequence  $t_n$ ,  $0 \geq t_n \geq t_0 + n\delta^*/2$  such that  $|\mathcal{A}(t)| \geq 1/4$  on  $[t_n, t_n + \delta^*] \cap I \supset [t_n, t_{n+1}] \cap I$ . Finally on  $I \cap [-1, 0]$  we have  $|\mathcal{A}(t)| \geq 1/4$ . This gives from (18) that for every  $t \in I \cap [-1, 0]$ :  $|\mathcal{B}(t)| \leq \varepsilon_1/16$ . Hence:

$$|\{\theta^* \geq 1\}| \leq \varepsilon_1/16 + \varepsilon_1/2 \leq \varepsilon_1.$$

Since  $(\theta^* - 1)_+ \leq 1$ , this gives that:

$$\int_{Q_1^*} (\theta^* - 1)_+^2 dx dz dt \leq \varepsilon_1.$$

We have for every  $t, x$  fixed:

$$\theta - \theta^*(z) = \int_0^z \partial_z \theta^* dz.$$

So:

$$\begin{aligned} (\theta - 1)_+^2 &\leq 2 \left( (\theta^*(z) - 1)_+^2 + \left( \int_0^z |\nabla \theta^*| dz \right)^2 \right) \\ &\leq \frac{2}{\sqrt{\varepsilon_1}} \int_0^{\sqrt{\varepsilon_1}} (\theta^* - 1)_+^2 dz + 2\sqrt{\varepsilon_1} \int_0^{\sqrt{\varepsilon_1}} |\nabla \theta^*|^2 dz. \end{aligned}$$

Therefore:

$$\int_{Q_1} (\theta - 1)_+^2 dx ds \leq C\sqrt{\varepsilon_1}.$$

□

## 6 Oscillation lemma

In the first technical lemma, we have established that if  $0 \leq \theta^* \leq 2$  and its energy or norm is very small, in  $B_4^*$ , then,  $\theta^* \leq 2 - \lambda$  in  $B_1$ , i.e., the oscillation of  $\theta$  actually decays. We want now to get rid of the "very small" hypothesis. This second lemma proves that if  $\theta^* \leq 0$  "half of the time" and it needs very little room,  $\delta$ , to go from  $\{\theta^* \leq 0\}$  to  $\{\theta \geq 1\}$ , it is because  $(\theta - 1)_+$  has very small norm to start with. This produces a dichotomy: or the support of  $\theta$  decreases substantially, or  $\theta$  becomes small anyway.

**Proposition 9** *There exists  $\lambda^* > 0$  such that for every solution  $\theta$  of (1) with  $v$  verifying (8), if:*

$$\begin{aligned} \theta^* &\leq 2 \quad \text{in } Q_1^* \\ |\{(t, x, z) \in Q_1^*; \theta^* \leq 0\}| &\geq \frac{1}{2}, \end{aligned}$$

then:

$$\theta^* \leq 2 - \lambda^* \quad \text{in } Q_{1/16}^*.$$

Note that  $\lambda^*$  depends only on  $N$  and  $C_v$  in (8).

**Proof.** For every  $k \in \mathbb{N}$ ,  $k \leq K_+ = E(1/\delta_1 + 1)$  (where  $\delta_1$  is defined in Lemma 8 for  $\varepsilon_1$  such that  $4C\sqrt{\varepsilon_1} \leq \varepsilon_0$ ,  $\varepsilon_0$  defined in Lemma 6), we define:

$$\bar{\theta}_k = 2(\bar{\theta}_{k-1} - 1) \quad \text{with} \quad \bar{\theta}_0 = \theta.$$

So we have:  $\bar{\theta}_k = 2^k(\theta - 2) + 2$ . Note that for every  $k$ ,  $\bar{\theta}_k$  verifies (1),  $\bar{\theta}_k \leq 2$  and  $|\{(t, x, z) \in Q_1^* \mid \bar{\theta}_k \leq 0\}| \geq \frac{1}{2}$ . Assume that for all those  $k$ ,  $|\{0 < \bar{\theta}_k^* < 1\}| \geq \delta_1$ . Then, for every  $k$ :

$$|\{\bar{\theta}_k^* < 0\}| = |\{\bar{\theta}_{k-1}^* < 1\}| \geq |\{\bar{\theta}_{k-1}^* < 0\}| + \delta_1.$$

Hence:

$$|\{\bar{\theta}_{K_+}^* \leq 0\}| \geq 1,$$

and  $\bar{\theta}_{K_+}^* < 0$  almost everywhere, which means:  $2^{K_+}(\theta^* - 2) + 2 < 0$  or

$$\theta^* < 2 - 2^{-K_+}.$$

And in this case we are done.

Else, there exists  $0 \leq k_0 \leq K_+$  such that:  $|\{0 < \bar{\theta}_{k_0}^* < 1\}| \leq \delta_1$ . From Lemma 8 and Lemma 6 (applied on  $\bar{\theta}_{k_0+1}$ ) we get  $(\bar{\theta}_{k_0+1})_+ \leq 2 - \lambda$  which means:

$$\theta \leq 2 - 2^{-(k_0+1)}\lambda \leq 2 - 2^{-K_+}\lambda,$$

in  $Q_{1/8}$ .

Consider the function  $b_3$  defined by:

$$\begin{aligned} \Delta b_3 &= 0 && \text{in } B_{1/8}^*, \\ b_3 &= 2 && \text{on the sides of the cube except for } z = 0 \\ b_3 &= 2 - 2^{-K_+} \inf(\lambda, 1) && \text{on } z = 0. \end{aligned}$$

We have  $b_3 < 2 - \lambda^*$  in  $B_{1/16}^*$ . And from the maximum principle we get  $\theta^* \leq b_3$ .  $\square$

## 7 Proof of Theorem 2.

We fix  $t_0 > 0$  and consider  $t \in [t_0, \infty[ \times \mathbb{R}^N$ . We define:

$$F_0(s, y) = \theta(t + st_0/4, x + t_0/4(y - x_0(s))),$$

where  $x_0(s)$  is solution to:

$$\begin{aligned} \dot{x}_0(s) &= \frac{1}{|B_4|} \int_{x_0(s)+B_4} v(t + st_0/4, x + yt_0/4) dy \\ x_0(0) &= 0. \end{aligned}$$

Note that  $x_0(s)$  is uniquely defined from Cauchy Lipschitz theorem. We set:

$$\begin{aligned} \tilde{\theta}_0^*(s, y) &= \frac{4}{\sup_{Q_4^*} F_0^* - \inf_{Q_4^*} F_0^*} \left( F_0^* - \frac{\sup_{Q_4^*} F_0^* + \inf_{Q_4^*} F_0^*}{2} \right), \\ v_0(s, y) &= v(t + st_0/4, x + t_0/4(y - x_0(s))) - \dot{x}_0(s), \end{aligned}$$

and then for every  $k > 0$ :

$$\begin{aligned} F_k(s, y) &= F_{k-1}(\tilde{\mu}s, \tilde{\mu}(y - x_k(s))), \\ \tilde{\theta}_k^*(s, y) &= \frac{4}{\sup_{Q_4^*} F_k^* - \inf_{Q_4^*} F_k^*} \left( F_k^* - \frac{\sup_{Q_4^*} F_k^* + \inf_{Q_4^*} F_k^*}{2} \right), \\ \dot{x}_k(s) &= \frac{1}{|B_4|} \int_{x_k(s)+B_4} v_{k-1}(\tilde{\mu}s, \tilde{\mu}y) dy \\ x_k(0) &= 0 \\ v_k(s, y) &= v_{k-1}(\tilde{\mu}s, \tilde{\mu}(y - x_k(s))) - \dot{x}_k(s), \end{aligned}$$

where  $\tilde{\mu}$  will be chosen later. We divide the proof in several steps.

**Step 1.** For  $k=0$ ,  $\tilde{\theta}_0$  is solution to (2) in  $[-4, 0] \times \mathbb{R}^N$ ,  $\|v_0\|_{BMO} = \|v\|_{BMO}$ ,  $\int v_0(s) dy = 0$  for every  $s$  and  $|\tilde{\theta}_0| \leq 2$ . Assume that it is true at  $k-1$ . Then:

$$\partial_s F_k = \tilde{\mu} \partial_s \tilde{\theta}_{k-1} - \tilde{\mu} \dot{x}_k(s) \cdot \nabla \tilde{\theta}_{k-1}.$$

So  $\tilde{\theta}_k$  is solution of (2) and  $|\tilde{\theta}_k| \leq 2$ . By construction, for every  $s$  we have  $\int_{B_4} v_k(s, y) dy = 0$  and  $\|v_k\|_{BMO} = \|v_{k-1}\|_{BMO} = \|v\|_{BMO}$ . Moreover we have:

$$\begin{aligned} |\dot{x}_k(s)| &\leq \int_{B_4} v_{k-1}(\tilde{\mu}(y - x_k(s))) dy \\ &\leq C \|v_{k-1}(\tilde{\mu}y)\|_{L^p} \\ &\leq C \tilde{\mu}^{-N/p} \|v_{k-1}\|_{L^p} \\ &\leq C_p \tilde{\mu}^{-N/p} \|v_{k-1}\|_{BMO}. \end{aligned}$$

So, for  $0 \leq s \leq 1$ ,  $y \in B_4$  and  $p > N$ :

$$|\tilde{\mu}(y - x_k(s))| \leq 4\tilde{\mu}(1 + C_p \tilde{\mu}^{-N/p}) \leq C \tilde{\mu}^{1-N/p}.$$

For  $\tilde{\mu}$  small enough this is smaller than 1.

**Step 2.** For every  $k$  we can use the oscillation lemma. If  $|\{\tilde{\theta}_k^* \leq 0\}| \geq \frac{1}{2}|Q_4^*|$  then we have  $\tilde{\theta}_k^* \leq 2 - \lambda^*$ . Else we have  $|\{-\tilde{\theta}_k^* \leq 0\}| \geq \frac{1}{2}|Q_4^*|$  and applying the oscillation lemma on  $-\tilde{\theta}_k^*$  gives  $\tilde{\theta}_k^* \geq -2 + \lambda^*$ . In both cases this gives:

$$|\sup \tilde{\theta}_k^* - \inf \tilde{\theta}_k^*| \leq 2 - \lambda^*.$$

and so:

$$|\sup_{Q_1^*} F_k^* - \inf_{Q_1^*} F_k^*| \leq (1 - \lambda^*/2)^k |\sup_{Q_1^*} F_0^* - \inf_{Q_1^*} F_0^*|.$$

**Step 3.** For  $s \leq \tilde{\mu}^{2n}$ :

$$\sum_{k=0}^n \tilde{\mu}^{n-k} x_k(s) \leq \tilde{\mu}^{2n} \sum_{k=0}^n \frac{\tilde{\mu}^{n-k}}{\tilde{\mu}^{-N/p}} \leq \frac{\tilde{\mu}^n}{2},$$

for  $\tilde{\mu}$  small enough. So

$$\left| \sup_{[-\tilde{\mu}^{2n}, 0] \times B_{\tilde{\mu}^n/2}^*} \theta^* - \inf_{[-\tilde{\mu}^{2n}, 0] \times B_{\tilde{\mu}^n/2}^*} \theta^* \right| \leq (1 - \lambda^*/2)^n.$$

This gives that  $\theta^*$  is  $C^\alpha$  at  $(t, x, 0)$ , and so  $\theta$  is  $C^\alpha$  at  $(t, x)$ .  $\square$

## A Proof of the De Giorgi isoperimetric lemma.

Let  $\omega \in H^1([-1, 1]^{N+1})$ . We denote:

$$\begin{aligned}\mathcal{A} &= \{x; \omega(x) \leq 0\} \\ \mathcal{B} &= \{x; \omega(x) \geq 1\} \\ \mathcal{C} &= \{x; 0 < \omega(x) < 1\},\end{aligned}$$

and

$$\chi = \mathbf{1}_{\{y_1 + s(y_1 - y_2)/|y_1 - y_2| \in \mathcal{C}\}}.$$

We have:

$$\begin{aligned}|\mathcal{A}||\mathcal{B}| &\leq \int_{\mathcal{A}} \int_{\mathcal{B}} (\omega(y_1) - \omega(y_2)) dy_1 dy_2 \\ &= \int_{\mathcal{A}} \int_{\mathcal{B}} \int_0^{|y_1 - y_2|} \nabla \omega(y_1 + s \frac{y_1 - y_2}{|y_1 - y_2|}) \cdot \frac{y_1 - y_2}{|y_1 - y_2|} ds dy_1 dy_2 \\ &= \int_{\mathcal{A}} \int_{\mathcal{B}} \int_0^{|y_1 - y_2|} \chi \nabla \omega(y_1 + s \frac{y_1 - y_2}{|y_1 - y_2|}) \cdot \frac{y_1 - y_2}{|y_1 - y_2|} ds dy_1 dy_2 \\ &\leq \int_{\mathcal{A}} \int_{\mathcal{B}} \int_0^\infty \chi \left| \nabla \omega(y_1 + s \frac{y_1 - y_2}{|y_1 - y_2|}) \right| ds dy_1 dy_2 \\ &\leq \int_{B_1} \int_{B_1} \int_0^\infty \chi \left| \nabla \omega(y_1 + s \frac{y_1 - y_2}{|y_1 - y_2|}) \right| ds dy_1 dy_2 \\ &\leq \int_{S_{N-1}} \int_{B_1} \int_0^\infty \frac{|\nabla \omega(y_1 + s\nu)|}{s^{N-1}} \mathbf{1}_{\{(y_1 + s\nu) \in \mathcal{C}\}} s^{N-1} d\nu ds dy_1 \\ &\leq C \int_{B_1} \int_{B_1} \frac{|\nabla \omega(y_1 + y_2)|}{|y_2|^{N-1}} \mathbf{1}_{\{y_1 + y_2 \in \mathcal{C}\}} dy_2 dy_1 \\ &\leq C \|\nabla \omega\|_{L^2} |\mathcal{C}|^{1/2}.\end{aligned}$$

□

## B Higher regularity

We give the proof of the following theorem.

**Theorem 10** *Let  $\theta$  be a solution of the quasi-geostrophic equation (2), (3) satisfying the regularity properties of Theorem 3:*

$$\begin{aligned}\theta &\in L^\infty(0, \infty; L^2) \cap L^2(0, \infty; H^{1/2}) \\ &\cap L^\infty([t_0, \infty[ \times \mathbb{R}^N) \cap C^\alpha([t_0, \infty[ \times \mathbb{R}^N),\end{aligned}$$

for every  $t_0 > 0$ . Then  $\theta$  belongs to  $C^{1,\beta}([t_0, \infty[ \times \mathbb{R}^N)$  for every  $\beta < 1$  and  $t_0 > 0$  and is therefore a classical solution.

**Proof:** We want to show the regularity at a fixed point  $y_0 = (t_0, x_0) \in ]0, \infty[ \times \mathbb{R}^N \subset \mathbb{R}^m$  where  $m = N + 1$ . Note that Changing  $\theta(t, x)$  by  $\theta(t, x - u(t_0, x_0)t) - \theta(t_0, x_0)$  if necessary, we can assume without loss of generality that  $\theta(y_0) = 0$  and  $u(y_0) = 0$ . The fundamental solution of:

$$\partial_t \theta + \Lambda \theta = 0$$

is the Poisson kernel:

$$P(t, x) = \frac{Ct}{(|x^2| + t^2)^{\frac{N+1}{2}}},$$

a homogeneous function of order  $-N$  if extended for  $t$  negative. the solution  $\theta$  of (2) can be represented as the sum of two terms.

$$\theta(t, x) = P(t, \cdot) * \theta_0 - g(t, x), \quad (19)$$

where:

$$\begin{aligned} g(t, x) &= \int_0^t \int_{\mathbb{R}^N} P(t - t_1, x - x_1) \operatorname{div}(u(t_1, x_1) \theta(t_1, x_1)) dt_1 dx_1 \\ &= \int_0^\infty \int_{\mathbb{R}^N} \nabla_x \tilde{P}(y - y_1) \cdot u(y_1) \theta(y_1) dy_1. \end{aligned}$$

In the last inequality, we denoted  $y = (t, x)$ ,  $\tilde{P}$  the extension of  $P$  for negative  $t$  with value 0, and we passed the divergence on  $\tilde{P}$ , which becomes a singular integral. The first term in (19) is smooth for  $t > 0$  and depends only on the initial data. We focus on the second one  $g(y)$ . We fix  $e \in \mathbf{S}_m$ , and estimate  $g(y_0 + he) - g(y_0)$  for  $h > 0$  in the standard way. We split the integral:

$$g(y_0) - g(y_0 + he) = \int_0^\infty \int_{\mathbb{R}^N} Q_0(y_0 - y_1, he) u(y_1) \theta(y_1) dy_1 \quad (20)$$

where:

$$Q_0(y, he) = \nabla_x \tilde{P}(y) - \nabla_x \tilde{P}(y + he),$$

into two parts, one on the ball  $B_{10h}$  centered to  $y_0$  and radius  $10h$ , and the second on the complement. The first part has no cancellation so we separate the integrals:

$$\begin{aligned} & \int_{B_{10h}} \mathbf{1}_{\{t_1 \geq 0\}} [\nabla_x \tilde{P}(y_0 - y_1) - \nabla_x \tilde{P}(y_0 + he - y_1)] u(y_1) \theta(y_1) dy_1 \\ &= \int_{B_{10h}} \mathbf{1}_{\{t_1 \geq 0\}} \nabla_x \tilde{P}(y_0 - y_1) u(y_1) \theta(y_1) dy_1 \\ & \quad - \int_{B_{10h}} \mathbf{1}_{\{t \geq 0\}} \nabla_x \tilde{P}(y_0 + he - y_1) u(y_1) \theta(y_1) dy_1. \end{aligned}$$

If  $\theta$  is  $C^\alpha$ ,  $\alpha > 0$ , from the Riesz transform  $u$  is also  $C^\alpha$ , and since  $\theta(y_0) = u(y_0) = 0$ , we have:

$$|u(y_1) \theta(y_1)| \leq \inf(|y_1 - y_0|^{2\alpha}, C). \quad (21)$$

So the first integral is convergent and bounded by  $Ch^{2\alpha}$ . To deal with the second one, notice that  $\nabla_x \tilde{P}$  have mean value zero on any slice  $t = C$  of  $B_{10h}$ , so we can add and subtract  $\theta(y_0 + he)u(y_0 + he)$ . We have:

$$|\theta(y_1)u(y_1) - \theta(y_0 + he)u(y_0 + he)| \leq Ch^\alpha |y_0 + he - y_1|^\alpha,$$

where we have used again that  $u(y_0) = \theta(y_0) = 0$ . Hence the integral is also convergent and bounded by  $Ch^{2\alpha}$ . This gives that the contribution of  $B_{10h}$  on (20) is smaller than  $Ch^{2\alpha}$ .

Outside of a neighborhood of size  $10h$  we use the cancelation of  $\nabla_x \tilde{P}$ . Up to Lipschitz regularity we just do:

$$\begin{aligned} & |\nabla_x [\tilde{P}(y_1 - y_0) - \tilde{P}(y_1 + he - y_0)]| \\ & \leq \frac{h}{|y_1 - y_0|^{m+1}}, \end{aligned}$$

and integrate against  $|u\theta|$  which verifies (21). This gives the bound:

$$\int_{|y_1 - y_0| \geq 10h} \frac{h}{|y_1 - y_0|^{m+1-2\alpha}} dy_1 \leq Ch^{2\alpha},$$

provided that  $2\alpha < 1$ . Altogether, this gives that if  $\theta \in C^\alpha$  with  $2\alpha < 1$ , then

$$|g(y_0) - g(y_0 + he)| \leq Ch^{2\alpha}.$$

Bootstrapping the argument gives that  $\theta$  is  $C^\alpha$  for any  $\alpha < 1$ .

To go beyond Lipschitz we consider a second order increment quotient:

$$Q_1(y, he) = |\nabla[\tilde{P}(y + he) + \tilde{P}(y - he) - 2\tilde{P}(y)]|.$$

We have:

$$g(y_0 + he) + g(y_0 - he) - 2g(y_0) = \int_{\mathbb{R}^m} \mathbf{1}_{\{t_1 \geq 0\}} Q_1(y_0 - y_1, he) u(y_1) \theta(y_1) dy_1.$$

Note that  $Q_1(y, he) = Q_0(y, he) - Q_0(y - he, he)$ , so for  $|y| < 20h$ , the local estimate of the previous argument together with the  $C^\alpha$  property of  $\theta$  and  $u$  gives:

$$\int_{B_{20h}} |Q_1(y_0 - y_1) u(y_1) \theta(y_1)| dy_1 \leq Ch^{2\alpha}.$$

For  $|y| > 20h$  and  $y$  not in the strip  $\mathcal{T}_h = [t_0 - h, t_0 + h] \times \mathbb{R}^N$ , we have:

$$|Q_1(y_0 - y_1, he)| \leq C \frac{h^2}{|y_0 - y_1|^{m+2}}.$$

and the corresponding integral:

$$\begin{aligned} & \int_{|y_0 - y_1| \geq 20h} \mathbf{1}_{\{y_1 \notin \mathcal{T}_h\}} |Q_1(y_0 - y_1) u(y_1) \theta(y_1)| dy_1 \\ & \leq C \int_{|y| \geq 20h} \frac{h^2}{|y|^{m+2}} (|y|^{2\alpha} \wedge 1) dy \\ & \leq Ch^{2\alpha}, \end{aligned}$$

whenever  $2\alpha < 2$ . It remains to control the contribution of the strip  $\mathcal{T}_h \setminus B_{20h}$ . The estimate on  $Q_0$  gives that on this strip:

$$|Q_1(y_1 - y_0, he)| \leq C \frac{h}{|y_1 - y_0|^{N+2}} \leq C \frac{h}{|x_1 - x_0|^{N+2}}.$$

Not that on  $\mathcal{T}_h \setminus B_{20h}$  we have  $|x_1 - x_0| \geq h$ . So the contribution of this strip is bounded by:

$$\begin{aligned} & \int_{t_0-h}^{t_0+h} \int_{|x_1-x_0| \geq h} \frac{h}{|x_1-x_0|^{N+2-2\alpha}} dx_1 dt_1 \\ & \leq C \frac{h^{2\alpha}}{h} \int_{t_0-h}^{t_0+h} dt_1 \\ & \leq Ch^{2\alpha}, \end{aligned}$$

whenever  $2\alpha < 2$ . That goes all the way to  $C^{1,\beta}$  for every  $\beta < 1$ .  $\square$

## C existence of solutions to (1)

In this appendix we sketch the existence theory of approximate solution of the equation (1) satisfying the truncated energy inequalities in the hypothesis of Theorem 1. We start by restricting the problem to  $B_1 \times [0, \infty]$  and adding an artificial diffusion term  $\varepsilon \Delta$ . We will use the eigenfunctions  $\sigma_k$  and eigenvalues  $\lambda_k^2$  of the Laplacian in  $B_1$ , that is:

$$\Delta \sigma_k + \lambda_k^2 \sigma_k = 0.$$

Note that  $\sigma_k^*(x, z) = \sigma_k(x) e^{-\lambda_k z}$  is the harmonic extension of  $\sigma_k$  for the semi-infinite cylinder  $Q_1 = B_1 \times [0, \infty]$  with data 0 in the lateral boundary, and:

$$\lambda_k \sigma_k(x) = \partial_\nu \sigma_k^*(x, 0),$$

where  $\partial_\nu$  is the normal derivative. Also:

$$\int_{Q_1} \lambda_k \sigma_k^2 dx dz = \int_{\partial Q_1} \sigma_k^* \partial_\nu \sigma_k^* dx = \int_{Q_1} |\nabla \sigma_k^*|^2 dx dz,$$

and thus formula is also correct for any series

$$g(x) = \sum f_k \sigma_k(x),$$

provided that  $\sum f_k^2 \lambda_k$  converges, i.e.,  $g \in H^{1/2}(B_1)$ .

We want to solve then in  $[0, \infty[ \times B_1$  the equation:

$$\partial_t \theta + \operatorname{div}(v\theta) = \varepsilon \Delta \theta - (-\Delta^{1/2})\theta, \quad (22)$$

where  $-\Delta^{1/2}\theta$  is understood as the operator that maps  $\sigma_k$  to  $\lambda_k \sigma_k = \partial_\nu \sigma_k^*$ .

For, say,  $v$  bounded and divergence free, this is straightforward using Galerkin method: Let us restrict (22) to  $\sigma_k$ , with  $1 \leq k \leq k_0$ , i.e. we seek a function:

$$\theta = \theta_{\varepsilon, k_0} = \sum_1^{k_0} f_k(t) \sigma_k(x)$$

that is a solution of the equation when tested against  $\sigma_k$ ,  $1 \leq k \leq k_0$ . The functions  $f_k$  are solutions to the following system of ODEs:

$$f'_k(t) = -[\varepsilon \lambda_k^2 + \lambda_k] f_k(t) + \sum_{l=1}^{k_0} a_{kl} f_l(t), \quad 1 \leq k \leq k_0,$$

with initial value:

$$f_k(0) = \int_{B_1} \theta_0(x) \sigma_k(x) dx,$$

where:

$$a_{kl} = \int_{B_1} v(t, x) \cdot \nabla \sigma_k(x) \sigma_l(x) dx.$$

Note that, since  $v$  is divergence free, the matrix  $a_{kl}$  is antisymmetric. This leads to the estimate:

$$\begin{aligned} \sum_{k=1}^{k_0} f_k^2(t_2) + \int_{t_1}^{t_2} \sum_{k=1}^{k_0} (\varepsilon \lambda_k^2 + \lambda_k) f_k^2(s) ds \\ = \sum_{k=1}^{k_0} f_k^2(t_1). \end{aligned}$$

In particular  $\theta_{\varepsilon, k_0}$  satisfies the energy inequality:

$$\begin{aligned} \|\theta_{\varepsilon, k_0}(t_2)\|_{L^2(B_1)}^2 + \int_{t_1}^{t_2} \left( \|\theta_{\varepsilon, k_0}(s)\|_{\dot{H}^{1/2}(B_1)}^2 + \varepsilon \|\theta_{\varepsilon, k_0}(s)\|_{\dot{H}^1(B_1)}^2 \right) ds \\ \leq \|\theta_{\varepsilon, k_0}(t_1)\|_{L^2(B_1)}^2. \end{aligned}$$

Notice also that what we call  $H^{1/2}(B_1)$  corresponds to the extension of  $\theta$  to the half cylinder, and such:

$$\|\theta\|_{\dot{H}^{1/2}(B_1)} \geq \|\theta\|_{\dot{H}^{1/2}(\mathbb{R}^N)}.$$

We now pass to the limit in  $k_0$  and denote  $\theta_\varepsilon$  the limit. If we test  $\theta_{\varepsilon, k_0}$  with a function  $\gamma \in L^\infty(0, T; L^2(B_1)) \cap L^2(0, T; H^1(B_1))$ , there is no problem in passing to the limit in the term:

$$\int_{t_1}^{t_2} \int_{B_1} (\nabla \gamma) v \theta_{\varepsilon, k_0} dx ds,$$

since  $\theta_{\varepsilon, k_0}$  converges strongly in  $L^2([0, T] \times B_1)$ . In particular, for  $\gamma = (\theta_\varepsilon - \lambda)_+ = \theta_{\varepsilon, \lambda}$  the term converges to:

$$\int_{t_1}^{t_2} \int_{B_1} \nabla[\theta_{\varepsilon, \lambda}]^2 v \, dx \, ds = 0,$$

provided that  $v$  is divergence free. This leads to the following corollaries:

**Corollary 11** *The function  $\theta_\varepsilon$  satisfies the hypothesis of Lemma 6 independently of  $\varepsilon$ , and therefore:*

$$\|\theta_\varepsilon(T)\|_{L^\infty(B_1)} \leq \frac{C}{T^{N/2}} \|\theta_\varepsilon(0)\|_{L^2(B_1)}.$$

**Corollary 12** *The same theorem is true for  $v \in L^2([0, T] \times B_1)$  independently of the  $L^2$  norm of  $v$ .*

**Proof.** We approximate  $v$  by a mollification  $v_\delta$ .  $\square$

**Corollary 13** *For  $\theta_0$  prescribed in  $L^2(\mathbb{R}^N)$ , the same result is true in  $[0, T] \times \mathbb{R}^N$ .*

**Proof.** We may rescale the previous theorems to the ball of radius  $M$  by applying them to  $\bar{\theta}(t, x) = M^{N/2}\theta(Mt, Mx)$ . This change preserves the  $L^2$  norm, and so we get:

$$\sup_{B_M} M^{N/2}\theta(s, y) \leq \frac{C}{(s/M)^{N/2}},$$

or

$$\sup_{B_M} \theta(s, y) \leq \frac{C}{(s)^{N/2}},$$

provided that  $v \in L^2(B_M)$  is divergence free. Then letting  $M$  go to infinity gives the result.  $\square$

**Final remark:** Since all the estimates are independent of  $\varepsilon$  we may let  $\varepsilon$  go to zero for the limit to be weak solution of the limiting equation, and satisfying the truncated energy inequalities.

Note also that the same approach can be taken for higher regularity. Indeed, the proof of higher regularity depends only on the truncated and localized energy inequality that is also satisfied by the  $\varepsilon$ -problem. We may then pass to the limit in  $\varepsilon$  and find a classical solution of the limiting problem.

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