Review Problem Set 1

FINAL EXAMINATION

Date: Wednesday, May 9
Time: 2 – 5 PM
Location: CPE 2.216

1. Find all solutions to the equation $A\,\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. Determine whether each of the following sets of vectors are linearly independent:

a) $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$

b) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 7 \\ 6 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 9 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$, and $\mathbf{v}_4 = \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix}$.

c) $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and $\mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}$.

d) $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 4 \\ 6 \\ 10 \end{pmatrix}$, and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ -6 \\ -9 \\ 15 \end{pmatrix}$.

3. Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the linear map defined by

$$T(\mathbf{x}) := \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \mathbf{x}.$$

Determine if T is injective, and if T is surjective.

4. Give examples of two matrices A and B such that

a) both AB and BA are defined and $AB = BA$.

b) both AB and BA are defined but $AB \neq BA$.

c) AB is defined but not BA.

5. Let $A, D \in \mathbb{R}^{3\times 3}$, where $D = \mathrm{diag}(2, 3, 5)$. Describe DA and AD in terms of (the rows and columns of) A.

6. a) Find the inverse of $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$.

b) Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

7. a) Let $E_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$. What row operation is effected via left multiplication by E_1?

b) Let $E_2 \in \mathbb{R}^{3\times 3}$ be the matrix such that left multiplication by E_2 results in the interchange of the first and second rows of any $3 \times n$ matrix. Find E_2.

8. Determine whether each of the following matrices is invertible:

a) $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

b) $\begin{bmatrix} 7 & 0 & 4 \\ 3 & 0 & 1 \\ 2 & 0 & 9 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

9. Suppose $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 2$, and $\det\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = 3$. Compute the following:

a) $\det\begin{bmatrix} a_{11} & a_{12} & 5a_{13} \\ a_{21} & a_{22} & 5a_{23} \\ a_{31} & a_{32} & 5a_{33} \end{bmatrix}$,

b) $\det\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$,

c) $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} - a_{11} & 3a_{22} - a_{12} & 3a_{23} - a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

d) $\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}\right)$.

10. a) Is $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 \leq 1 \right\}$ a vector subspace of $\mathbb{R}^2$? Explain.

b) Is $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; xy \geq 0 \right\}$ a vector subspace of $\mathbb{R}^2$? Explain.

c) Is $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x + y + z = 1 \right\}$ a vector subspace of $\mathbb{R}^3$? Explain.

d) Is $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x + y + z = 0 \right\}$ a vector subspace of $\mathbb{R}^3$? Explain.

11. a) Does the set $\mathbb{R}[x]_5$ of all polynomials in the variable x of degree 5 form a vector space under the usual operations?

b) Does the set $\mathbb{R}[x]_{\leq 5}$ of all polynomials in the variable x of degree at most 5 form a vector subspace of $\mathbb{R}[x]$?

c) Does the set $\mathbb{R}[x]_{\geq 5}$ of all polynomials in the variable x of degree at least 5 form a vector subspace of $\mathbb{R}[x]$?

d) Does $V := \{ p(x) \in \mathbb{R}[x] \mid p(0) = 0 \}$ form a vector subspace of $\mathbb{R}[x]$?

12. Given that A can be row-operated to B, find a basis for $\mathrm{Col}(A)$, where

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

13. Define $T := \mathbb{R}[\,x\,]_{\leq 2} \longrightarrow \mathbb{R}^2$ by $T(p(x)) := \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$.

 a) Prove that T is a linear map.

 b) Find a basis for the kernel of T. Is T injective?

 c) Show that T is surjective.

14. a) Show that

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$$

 form a basis for $\mathbb{R}^n$.

 b) Show that $1,\, x,\, x^2,\, \cdots,\, x^n \in \mathbb{R}[\,x\,]_{\leq n}$ form a basis for $\mathbb{R}[\,x\,]_{\leq n}$.

15. Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 7 \\ 5 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 8 \\ 7 \\ 6 \\ 5 \end{pmatrix}.$$

 Find a basis for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

16. Let $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $\mathcal{B} = \{\, \mathbf{b}_1,\, \mathbf{b}_2 \,\}$.

 a) Let $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Find $M_{\mathcal{B}}(\mathbf{v})$.

 b) Given $M_{\mathcal{B}}(\mathbf{w}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, find $\mathbf{w}$.

 c) Let $\mathcal{E} = \{\, \mathbf{e}_1,\, \mathbf{e}_2 \,\}$ be the standard basis for $\mathbb{R}^2$, i.e. $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the change-of-basis matrix $M_{\mathcal{B}}(\mathcal{E})$.

 d) Given $M_{\mathcal{B}}(\mathbf{u}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $M_{\mathcal{E}}(\mathbf{u})$.

17. Exhibit a basis for the vector space $\mathbb{R}^{m\times n}$ of $m \times n$ matrices. Find $\dim(\mathbb{R}^{m\times n})$.

18. Let $V = \left\{ \begin{pmatrix} 3x + 6y - z \\ 6x - 2y - 2z \\ -9x + 5y + 3z \\ -3x + y + z \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x, y, z \in \mathbb{R} \right\}$. Find a basis for V and find $\dim V$.

19. Find all eigenvalues of $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$, and a basis for each of its eigensubspaces.

20. a) Can 0 be an eigenvalue of a matrix? If so, give an example of a matrix that admits 0 as an eigenvalue.

 b) Can $\mathbf{0}$ be an eigenvector of a matrix? If so, give an example of a matrix that admits $\mathbf{0}$ as an eigenvector.

21. Let $A = \begin{bmatrix} 1 & 3 & 8 & 4 & -2 \\ 0 & -1 & 4 & -2 & 5 \\ 0 & 0 & -2 & 7 & 6 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$. Is A diagonalizable? What are the eigenvalues of A?

22. Suppose $A \in \mathbb{R}^{n\times n}$. What is the characteristic polynomial of A? What is the degree of the characteristic polynomial of A? What are the roots of the characteristic polynomial of A?

23. Show that two similar matrices have the same eigenvalues.

24. Diagonalize $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

25. Diagonalize $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$.

26. Let $V := \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the vector space of (arbitrary) $\mathbb{R}$-valued functions defined on $\mathbb{R}$. Let $W := \{f \in V \mid f(0) = 1\}$. Is W a subspace of V? Justify.

27. (10 pts) Compute the determinant of the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

28. (10 pts) Prove that the image of a linear map is always a vector subspace of the codomain vector space.

29. (10 pts) Fine the dimension of the subspace of $\mathbb{R}^4$ spanned by the following vectors:

$$\begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ -5 \\ 5 \\ -5 \end{bmatrix}.$$

30. (10 pts) Find a basis for the kernel of the linear map $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by:

$$f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} := \begin{bmatrix} -2 & 2 & 2 & 1 \\ 1 & -1 & 0 & 0 \\ -5 & 5 & 4 & 2 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

31. Let $L : V \longrightarrow W$ be an injective linear map, and $\mathcal{S} \subset V$ be a linearly independent subset of V. Prove that

$$L(\mathcal{S}) := \{\ L(\mathbf{s}) \in W \mid \mathbf{s} \in \mathcal{S}\ \}$$

is a linearly independent subset of W.

(Warning: There are no restrictions on the dimensions of V and W, and the number of elements of S; i.e. any of them may be infinite.)

32. Let $L : V \longrightarrow W$ be a surjective linear map, and $\mathcal{S} \subset V$ be a spanning subset of V. Prove that

$$L(\mathcal{S}) := \{\ L(\mathbf{s}) \in W \mid \mathbf{s} \in \mathcal{S}\ \}$$

is a spanning subset of W.

(Warning: There are no restrictions on the dimensions of V and W.)

33. Give an example of a linear map between two infinite-dimensional vector spaces which is injective but not surjective.

(Recall that a linear map between two finite-dimensional vector spaces of the same dimension is injective if and only if it is surjective. This exercise shows this fails in infinite dimensions.)

Review Problem Set 1

34. Let $A \in \mathbb{F}^{n\times n}$ and $L : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ be the linear map given by $L(\mathbf{v}) := A \cdot \mathbf{v}$, for each $\mathbf{v} \in \mathbb{F}^n$. Prove that the following are equivalent:

    a) L is injective.

    b) L is surjective.

    c) L is an isomorphism.

    d) A is an invertible matrix.

35. Let $L : V \longrightarrow W$ be an injective linear map. Prove that V is isomorphic to image(L).

    Remark: This exercise justifies the intuition that an injective linear map $L : V \longrightarrow W$ "embeds" a copy of V into W, or that the subspace image(L) of W is a copy of V inside W.

36. Let $A \in \mathbb{F}^{m\times n}$ and $L : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ be the linear map given by $L(\mathbf{v}) := A \cdot \mathbf{v} \in \mathbb{F}^m$, for each $\mathbf{v} \in \mathbb{F}^n$. Prove that $\dim(\operatorname{image} L) = \operatorname{rank}(A)$.

37. a) Prove that the dimension of $\mathbb{R}[\,x\,]_{\leq n}$ is $n + 1$.

    b) Prove that $\mathbb{R}[\,x\,]$ is infinite-dimensional.

38. Let V be a finite-dimensional vector space with $n := \dim(V) < \infty$. Let $\mathcal{B} = (\,\mathbf{b}_1, \ldots, \mathbf{b}_n\,)$ be an ordered basis for V. Define $L : V \longrightarrow \mathbb{F}^n : \mathbf{v} \mapsto M_{\mathcal{B}}(\mathbf{v})$. Prove that L is an isomorphism.

39. Let $f : V \longrightarrow \mathbb{F}$ be a non-constant linear map, with $\dim V < \infty$. Find the dimension of the image of V. Find the dimension of the kernel of f in terms of $\dim V$.

40. What is $\dim_{\mathbb{R}}(\mathbb{C}[\,x\,]_{\leq 2})$? Prove your answer.

41. Let V be a vector space and $\mathcal{S} \subset V$ is a linearly independent subset of V. Suppose $\mathbf{v} \notin \operatorname{span}\mathcal{S}$. Prove that $\mathcal{S} \cup \{\mathbf{v}\}$ is still linear independent. (Warning: Do NOT assume $\mathcal{S}$ is a finite set.)

42. Prove that $f : \mathbb{R} \longrightarrow \mathbb{R} : x \longmapsto 2x + 7$ is not linear.