The Chern-Simons TQFT

The general subject of this workshop is Knot Invariants, in particular Khovanov homology. In the last two talks we have heard something about how Modular Tensor Categories, 2+1 dimensional TQFT's, and 2D Modular Functors are related, and how they give rise to knot invariants. The subject of this talk is a particular (family) of 2+1D TQFT's, which were first used by Witten (1989) to study knot invariants. The flavour of this talk will be alltogether more physical than the preceding talks, and, most likely, those that are to follow. In particular, I shall make free use of such tools as the Path Integral, which physicists use without blinking, but which cause our mathematical eyes to water.

The organisation of this talk will be as follows:

1. I shall review the concept of a TQFT.
2. I shall introduce "Classical Chern-Simons".
3. I shall "Quantise" the classical theory by means of the path integral.
   - Time permitting, I shall discuss the "weak-coupling" limit.
4. I shall construct the TQFT via "canonical quantisation".
5. Using the TQFT structure, I shall "solve" the field theory; in particular, I shall show how the Jones polynomial arises in this context, as well as how the theory calculates a 3-mfld invariant.

The talk will follow Witten's paper fairly closely in parts, with additional material coming from Bakalov & Kirillov's book.

TQFT

Despite their name, TQFT's have more to do with mathematics than with physics. They first arose in an attempt to axiomatise QFT (Atiyah) in a rather rough fashion, and have since taken on a life of their own.

An $n+1$ dimensional TQFT, then, is a functor from the bordism category of (smooth, & oriented, compact) n-manifolds to the category of vector spaces (ab. groups, certain categories ...), ~~...~~ More precisely:

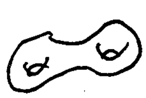
- To every (compact, smooth, oriented) n-fold it associates a vector space:

 $\longrightarrow V$

- To the disjoint union of two n-folds, it associates the tensor product of their associated vector spaces:

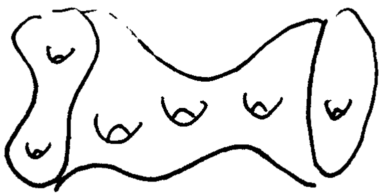
 $\longrightarrow V$

 $\longrightarrow W$

$\Rightarrow$  $\amalg$  $\longrightarrow V \otimes W$.

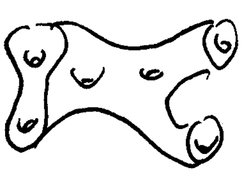
- If an n-fold M is associated with V, then M with the opposite orientation is associated to V^*.

• To every bordism between n-folds, it associates a vector-space morphism = linear map ~~V→W~~ from the "in-going" to the "outgoing" bdd.

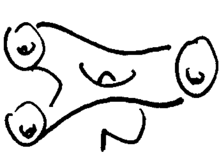
 $\rightsquigarrow \quad V \to W$

[This is sometimes stated as follows: If M is an oriented $n+1$ fold, ~~then~~ with bdd ∂M, then the TQFT assigns to M an element in the vector space assigned to ∂M. This agrees with the above, as, if $\partial M = \Sigma^{op} \sqcup \Xi$, Σ "ingoing", Ξ "outgoing", then $\partial M \rightsquigarrow V^* \otimes W$ if $\Sigma \rightsquigarrow V$, $\Xi \rightsquigarrow W$, and an element of $V^* \otimes W$ is precisely a linear map $V \to W$.]

• (Gluing) If M is an n-fold, with $\partial M = \Sigma_1^{op} \sqcup \Sigma_2$, N an n-fold with $\partial N = \Sigma_2^{op} \sqcup \Sigma_3$, then the morphism assigned to $M \cup_{\Sigma_2} N$ must be the composition of morphisms assigned to M, N.

 $\rightsquigarrow \quad V \to W$

M

 $\rightsquigarrow \quad W \to Z$

N

$M \cup_{\Sigma_2} N$ $\longrightarrow$

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend right] & W \arrow[r] & Z \end{tikzcd}$$
$$\parallel$$
$$V \to Z$$

[In order to make gluing easier to define, bordisms are usually assumed to have collar nbhds of the bdd.]

- This is all up to homeomorphism:
  - If $\Sigma_1 \cong \Sigma_2$, two n-folds, and $\Sigma_{1,2} \to V_{1,2}$, then the TQFT assigns an isomorphism $\varphi: V_1 \to V_2$.
  - If $M_1 \cong M_2$, $n+1$-folds, $\partial M_1 = \Sigma_{1a}^{op} \sqcup \Sigma_{1b}$, $\partial M_2 = \Sigma_{2a}^{op} \sqcup \Sigma_{2b}$, & M_1: $\Sigma_{1a} \to V_{1a}$, $\Sigma_{1b} \to V_{1b}$, $\Sigma_{2a} \to V_{2a}$, $\Sigma_{2b} \to V_{2b}$, $\varphi_{a,b}: V_{1a,b} \to V_{2a,b}$ isos, then

$$\begin{tikzcd} V_{1a} \arrow[r, "M_1"] \arrow[d, "\varphi_a"'] \arrow[dr, phantom, "\circlearrowright"] & V_{1b} \arrow[d, "\varphi_b"] \\ V_{2a} \arrow[r, "M_2"'] & V_{2b} \end{tikzcd}$$

- Normalization axioms:
  - Σ an n-fold, I the unit interval, then
    $\Sigma \times I \to \mathrm{Id}: V \to V$ if
    $\Sigma \to V$.

- $S^n \to k$

  $\emptyset \to k$

  $B^{n+1} \to k \xrightarrow{1} k$.

- Everything must be compatible.

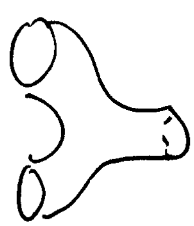
An Example (1+1).

Let us consider the 1+1 D case. There is only one connected, compact 1-fold, S^1, to which we assign

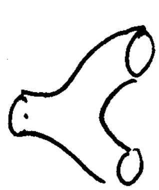
 $\to V$.

Then

 $\to Tr: V \to \mathbb{C}$

 $\to m: V \otimes V \to V$

commutative.

 $\to \Delta: V \to V \otimes V$

co-mult

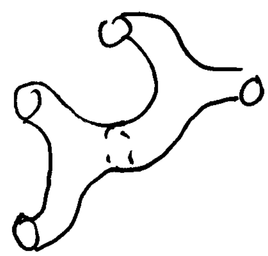
 $\to e: \mathbb{C} \to V$ unit

So you get a Fröbenius algebra.

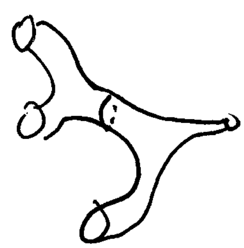
You prove things like ass. as follows

~~Ⓐ m(v, w~~

$m(\cdot, m(\cdot,\cdot)) \rightsquigarrow$



$\cong$



$\rightsquigarrow m(m(\cdot,\cdot),\cdot)$.

~~In~~ We will construct a decorated 2+1 TQFT.

Our ~~In~~ surfaces will have marked points, ~~with~~ an ~~in~~ or decorated in certain ways, and the bordisms will have embedded links, and lines from the incoming marked pts to the outgoing pts. This will be explained in detail later.

Classical Chern Simons

Let us now write down a field theory associated to a 3-fold X. We wish the theory to be "generally covariant", that is, depends only on the topology of the 3-fold, and nothing more (in fact, it will be defined on a smooth 3-fold, but as all 3-folds admit a unique smooth structure, this will be the same as defining a theory that depends only on the topology of X).

The standard ~~for~~ prescription for formulating a generally covariant theory is to start with a Riemannian ~~metric~~ structure, and then integrate over all such. We shall proceed rather differently.

Endow $X \times G \to X$ (the trivial principal G-bundle over X, G cpt) with a connection $\nabla = d + A$, where A is an $Ad(\mathfrak{g})$ valued 1-form. ~~We define~~ ~~to~~ The dynamical variable will be the connection, and the Chern-Simons action is defined to be

$$\mathcal{L} = \frac{k}{4\pi} \int_X \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

The integrand is called the Chern-Simons three form, and is a combination of all possible ways to form a three form from the connection. ~~Fe.~~

Let us examine how $\mathcal{L}$ changes under gauge-transformations: The group $\hat{G}$ of cts maps $M \to G$ is not connected (for instance $\pi_3(G) \simeq \mathbb{Z}$ for $\forall$ cpt simple G). Now, $\mathcal{L}$ is invariant under transformations in the id. cpt of $\hat{G}$, but for all other such,

$\mathcal{L} \mapsto \mathcal{L} + \text{const} \cdot m$.

In order to define a sensible quantum theory, we require $\mathcal{L}$ changes by multiples of 2π (as one takes $e^{i\mathcal{L}}$)

WHICH FIXES k TO BE AN INTEGER FOR G A MATRIX GROUP, AND "Tr" REALLY MEANING TRACE. FOR ANY OTHER GROUP, WE WILL NORMALISE TRACE S.T. k IS INTEGRAL.

~~WHAT ARE GENERALLY COVARIANT OBSERVABLES~~

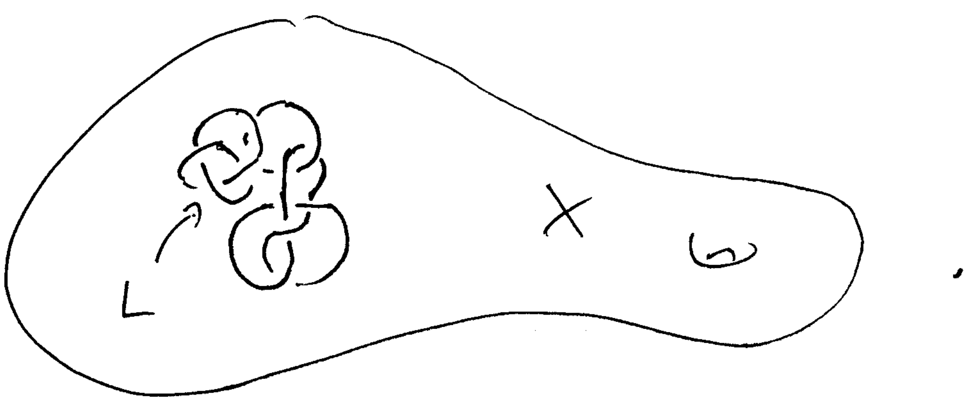
- QUANTISATION VIA THE PATH INTEGRAL

FEYNMANN'S QUANTISATION PRESCRIPTION TELLS US, THAT TO COMPUTE THE (UNNORMALISED) EXPECTATION OF OBSERVABLES $\mathcal{O}_1, \mathcal{O}_2 \ldots$, WE MUST COMPUTE

$$\int \mathcal{D}\mathcal{A} \exp(i\mathcal{L})\, \mathcal{O}_1 \mathcal{O}_2 \ldots \quad .$$

IN OUR CASE, THE INTEGRAL IS TAKEN OVER ALL CONNECTIONS ON G, MODULO GAUGE ~~EQUIVAL.~~ TRANSFORMATIONS, AND $\mathcal{L}$ IS THE CHERN-SIMONS ACTION. WE HAVE YET TO FIND APPROPRIATE GENERALLY COVARIANT OBSERVABLES.

LET US NOW SUPPOSE WE HAVE A LINK L EMBEDDED IN X:



ENDOW L WITH AN ORIENTATION, AND COLOUR EACH COMPONENT i WITH A REPRESENTATION R_i OF G. OUR OBSERVABLE WILL THEN BE THE TRACE (IN THE R_i's) OF THE HOLONOMY AROUND L (THIS IS USUALLY CALLED A "WILSON LINE").

DEFINE, THEN

$$Z(\cancel{X}; R_i, C_i) = Z(X; R, L)$$

$$= \int \mathcal{D}\mathcal{A} \exp(i\mathcal{L}) \prod_i \mathrm{Tr}_{R_i} \mathrm{Hol}_{C_i}(A).$$

THESE ARE MANIFESTLY INVARIANTS OF THE LINK IN X.

WEAK COUPLING LIMIT

WHEN WITTEN INTRODUCED THE CHERN-SIMONS TQFT, AS HE MADE AN EXTENDED DIVERSION INTO THE SO-CALLED WEAK-COUPLING LIMIT, BOTH IN ORDER TO UNDERSTAND THE QUALITATIVE BEHAVIOUR OF THE THEORY, AND AS A CHECK TO MAKE SURE THE THEORY, IN FACT, MAKES SENSE. IN THIS CONTEXT, THE WEAK COUPLING LIMIT IS THE LARGE k LIMIT.

I DO NOT PROPOSE TO DISCUSS THE WEAK-COUPLING LIMIT IN ALL OF ITS GORY DETAILS, BUT PERHAPS I SHALL TAKE A MOMENT TO POINT OUT ITS SALIENT FEATURES.

LET US FIRST EXAMINE THE THEORY IN THE ABSENCE OF KNOTS. WHEN k IS LARGE, THE PATH INTEGRAL LOCALISES AT THE (GAUGE EQUIVALENCE CLASSES) OF FLAT CONNECTIONS, SO LOOKS LIKE

$$Z = \sum_\alpha \mu(A^\alpha),$$

WHERE THE A^α RUN OVER ALL GAUGE EQUIVALENCE CLASSES OF FLAT CONNECTIONS, AND $\mu(A^\alpha)$ IS THE ST. PHASE EVALUATION

OF THE PATH INTEGRAL NEAR A^{α}. SOME WORK SHOWS

$$\mu(A^{\alpha}) = \exp(ik\, I(A^{(\alpha)})) \cdot \frac{\det(\Delta)}{\sqrt{\det(L_-)}} ,$$

WHERE

$$I(A^{\alpha}) = \frac{1}{4\pi} \int_M \mathrm{Tr}\left(\tfrac{2}{3} A^{\alpha} \wedge A^{\alpha} \wedge A^{\alpha} + \tfrac{2}{3} A^{\alpha} \wedge dA^{\alpha}\right),$$

Δ IS THE LAPLACIAN,

AND

$L_{\alpha}^{*} = D* + *D$, WITH D THE FLAT CONNECTION $(D = d + A^{\alpha})$.

L_- IS L RESTRICTED TO ODD FORMS, AND THE DETERMINANTS ARE RENORMALISED.

IT TURNS OUT

$$\left| \frac{\det(\Delta)}{\det(L_-)} \right|$$

IS THE RAY-SINGER ANALYTIC TORSION OF A^{α}, SO THAT ~~HENCE~~ $|\mu(A^{\alpha})|$ IS A TOP. INVARIANT. HOWEVER, THE PHASE TURNS OUT TO BE MORE DELICATE. THE QUESTION BOILS DOWN TO HOW THE PHASE OF $\frac{1}{\sqrt{\det L_-}}$ BEHAVES (EVERYTHING ELSE BEING REAL); AND ONE FINDS

$$\frac{1}{\sqrt{\det L_-}} = \frac{1}{|\sqrt{\det L_-}|} \cdot \exp\left(\frac{i\pi}{2}\, \eta(A^{(\alpha)})\right),$$

WHICH IS NOT TOP. INVARIANT.

Some fiddling (adding an appropriate counterterm) shows that you can, in fact, derive ~~a~~ Z as an invariant.

• Including knots.

The situation becomes more delicate with the introduction of knots. The important conclusion reached in examining this case is that knots should come with a framing (a choice of normal vector field (up to eq.)). This seems rather inconvenient — the Jones polynomials, for instance, do not require a choice of framing. There are several remarks to make:

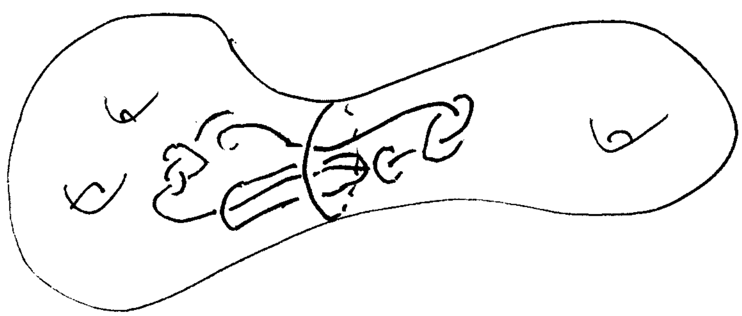
- First, for knots in S^3 there is a canonical choice of framing.
- Second, a choice of framing ~~m~~ will in fact turn out to be a useful feature in the sequel.
- ~~A~~ Third, there are ~~several equiv.~~ other, more computationally useful ways to "frame" the knot (see, for instance, ~~Gen~~ Freed-Gompf).

Canonical Quantisation

In order to understand the Chern-Simons QFT, and "solve" it, we need to realise it explicitly as a TQFT. A first step would be to understand what we assign to the boundaries. Before doing so, let us take a step back, and try to understand what the nature of the TQFT is. We are dealing with a QFT in three dimensions, so our strategy will be to chop up the 3-folds into pieces, and see the QFT as a 2+1-D TQFT, and use gluing to understand the invariants we wish on the total space.

A PLAIN-VANILLA 2+1 D TQFT DESCRIPTION

HOWEVER, IT WILL BE AN INSUFFICIENT FOR OUR PROBLEM: WE HAVE FRAMED LINKS EMBEDDED IN OUR 3-FOLD, AND THE QUANTITIES OF INTEREST DEPEND ON THESE LINKS. SLICING THE 3-FOLD ARBITRARILY WILL GIVE US A 3-FOLD WITH EMBEDDED ARCS AND CIRCLES, ALL FRAMED AND COLOURED BY REPRESENTATIONS, WITH A MARKED SURFACE ON THE BDD, EACH MARKED POINT COLOURED BY A REP^N OF G:



THUS WE EXPECT TO HAVE AS BDD SURFACES, SURFACES WITH ~~...~~ MARKED POINTS COLOURED BY REP^NS OF G, AND AS BORDISMS 3-FOLDS WITH BDD, WITH EMBEDDED ARCS JOINING THE MARKED POINTS, COLOURED BY REP^NS OF G (AND COLOURING AGREEING W/ THE BDD), AND EMBEDDED CIRCLES COLOURED BY REPS OF G.

AS A FIRST STEP, LET US TRY TO DETERMINE WHAT THE SPACE IS ASSOCIATED WITH A SURFACE Σ WITH NO MARKED POINTS. TO THAT END, WE WILL FOLLOW WITTEN AND CANONICALLY ~~...~~ QUANTISE THE CHERN-SIMONS TQFT ON $\Sigma \times \mathbb{R}$. ROUGHLY SPEAKING, CANONICAL QUANTISATION GIVES A WAY TO ASSOCIATE WITH A CLASSICAL SYSTEM (WITH PHASE SPACE AND SYMPLECTIC FORM) A QUANTUM SYSTEM, WHERE THE PHASE SPACE IS REPLACED BY A HILBERT SPACE (USUALLY THE SPACE OF FN'S ON THE ORIGINAL

PHASE SPACE), AND THE OBSERVABLES (REAL FUNCTIONS ON THE PHASE SPACE) (13)
ARE REPLACED BY OPERATORS ON H SATISFYING THE CANONICAL COMM. RELATIONS, i.e. IF f, g ARE OBSERVABLES OF THE CLASSICAL SYSTEM, AND $\{f, g\} = h$, THEN $[\hat{f}, \hat{g}] = i\hbar \hat{h}$ (CONSTANTS MAY BE OUT).

IN OUR CASE THE PHASE SPACE IS SPACE OF CONNECTIONS ON $X \times G$. HOWEVER, THEY ARE SUBJECT TO CERTAIN CONSTRAINTS, AND, IMPOSING THESE, WE REDUCE THE PHASE SPACE TO THE MODULI SPACE OF FLAT CONNECTIONS MODULO GAUGE TRANSFORMS ON Σ, $\mathcal{M}$. $\mathcal{M}$ CARRIES A CANONICAL SYMPLECTIC FORM, BUT NO NATURAL KÄHLER STRUCTURE. THIS IS PROBLEMATIC, AS ~~THE~~ $\mathcal{M}$ IS OF FINITE VOLUME, AND THE CANONICAL QUANTIZATION OF SYSTEMS WITH FINITE VOLUME PHASE SPACE HAS NO KNOWN GENERAL SOLUTION, BUT CAN BE DONE IF THE SPACE IS KÄHLER. IN THIS CASE WE CHANGE VARIABLES FROM "POSITION" q AND "MOMENTUM" p, TO $z \sim p + iq$, $\bar{z} = p - iq$, AND LET $\mathcal{H}$ BE THE (FINITE DIM'L) SPACE OF HOL. FNS ON $\mathcal{M}$.

IN OUR CASE WE ARE SAVED BY THE FACT THAT A CHOICE OF COMPLEX STRUCTURE J ON Σ GIVES $\mathcal{M}$ A KÄHLER STRUCTURE, ALLOWING US TO ASSOCIATE $\mathcal{H}^J$, THE SPACE OF HOL. FUNCTIONS ON $\mathcal{M}_J$. A MORE CONCRETE DESCRIPTION MAY BE GIVEN: $\mathcal{M}_J$ PARAMETERISES THE SPACE OF HOLOMORPHIC VECTOR BUNDLES ON Σ, AND SO CAN BE THOUGHT OF PARAMETERISING A FAMILY OF $\bar{\partial}$ OPERATORS (HOL. CONN. OF THE BUNDLES). ~~THE FIRST CHERN CLASS~~
ONE MAY FORM THE DETERMINANT LINE BUNDLE $\mathrm{Det}(\bar{\partial})$ AND THEN THE SPACE $\mathcal{H}^J = \mathcal{O}(\mathrm{Det}(\bar{\partial})^{\otimes k})$ (~~BUT~~ NOT PRECISELY TRUE IF $G \neq SU(N)$), AND THE CANONICAL SYMPL. FORM IS $\omega = c_1(\mathrm{Det}(\bar{\partial})^{\otimes k})$.

We would like to remove the dependence on the choice of J. For each J we get a vector space $\mathcal{H}^J$ ~~[illegible]~~, which varies holomorphically with J, and thus we get a vector bundle over the moduli space of complex Riemann surfaces. Saying that the $\mathcal{H}$ is independent of J amounts to asserting this bundle have a canonical flat connection. This turns out to be true projectively.

The spaces $\mathcal{H}^J$ ~~have~~ occur in conformal field theory where they are the "space of conformal blocks". The association $\Sigma \to \hat{\mathcal{H}}_\Sigma$ is called a "modular functor".

If one extends this ~~quantiza~~ reasoning to surfaces with marked points, one sees

$$\Sigma_{P_i, R_i} \longrightarrow \hat{\mathcal{H}}_{\Sigma, P_i, R_i}$$

~~where~~ the space of conformal blocks at level k with primary fields transforming as R_i inserted at P_i.

For the present, we will be content to know:

$$S^2_{\emptyset,\emptyset} \longrightarrow 1d \text{ space}$$

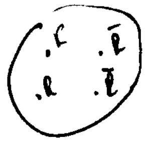
$$S^2_{P,R} \longrightarrow \begin{cases} 1d & R \text{ trivial} \\ 0d & \text{otherwise} \end{cases}$$

$$S^2_{P_1,P_2;R_1,R_2} \longrightarrow \begin{cases} 1d & \text{if } R_1 = R_2^* \\ 0d & \text{otherwise} \end{cases}$$

$S^2_{P_1,P_2,P_3;R_1,R_2,R_3} \to \dim \mathcal{H} = N_{R_1 R_2 R_3}$, the number proposed by Verlinde.

For an arbitrary # of pts on S^2, the spaces may be determined by the N_{ijk}

In particular, suppose we have



,

with $R \otimes \bar{R} = \sum_{i=1}^{s} E_i$, E_i irreps, distinct.

Then for large k, $\dim \mathcal{H} = s$. If $G = SU(N)$, R is the defining rep, $s = 2$ and $\dim \mathcal{H} = 2$ (unless $k=1$, when $\dim \mathcal{H} = 1$).

Calculations (everything done in BB framing)

Let us now use the TQFT structure along with the answers in the previous section to deduce

$$Z(M_1 \# M_2) \cdot Z(S^3) = Z(M_1) \cdot Z(M_2),$$

where M_1, M_2 may have knots included.

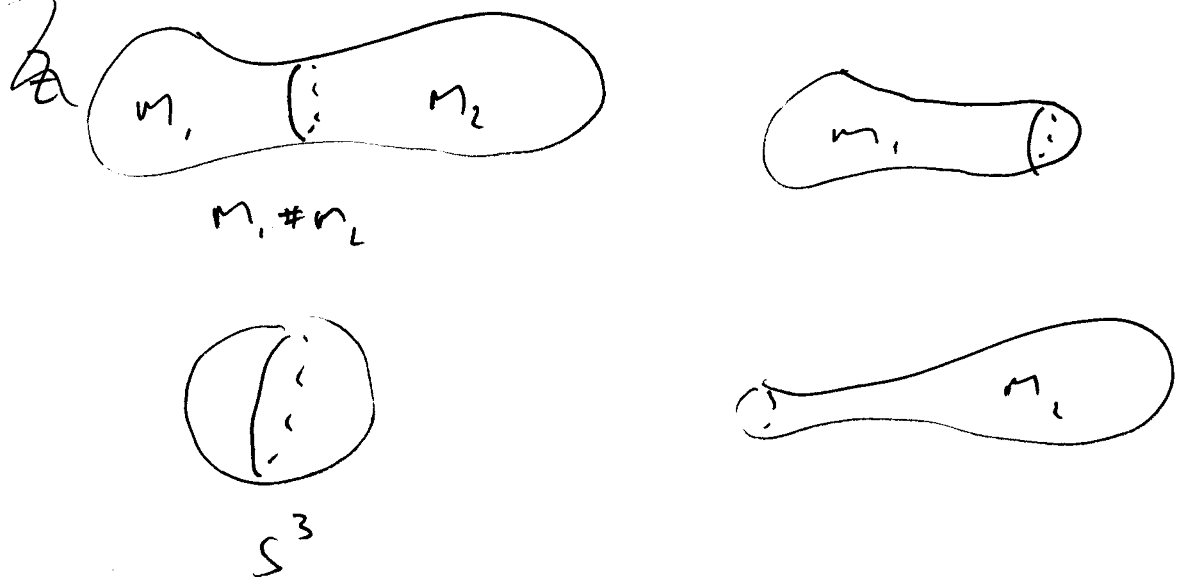
Let us cut a 3 ball out of M_1, M_2 such that the ball misses all included knots. The ~~product~~ vector space associated to the resulting boundary spheres must be 1-D, and $M_{1,2} \setminus B_{1,2}$ must give vectors $\xi_{1,2}$ in these. Then

$$Z(M_1 \# M_2) = \langle \xi_1, \xi_2 \rangle.$$

On the other hand, ∂B_i gets associated to the same 1-D space, and B_i gives a vector v_i in it. Gluing the B_i shows $\langle v_1, v_2 \rangle = Z(S^3)$. Likewise, $Z(M_1) = \langle \xi_1, v_1 \rangle$, $Z(M_2) = \langle v_2, \xi_2 \rangle$, and, as the space is 1-D, we have

$$\langle \xi_1, \xi_2 \rangle \langle v_1, v_2 \rangle = \langle \xi_1, v_1 \rangle \langle v_2, \xi_2 \rangle \quad \square$$

Thus



Thus

$$\frac{Z(M_1 \# M_2)}{Z(S^3)} = \frac{Z(M_1)}{Z(S^3)} \, \frac{Z(M_2)}{Z(S^3)} .$$

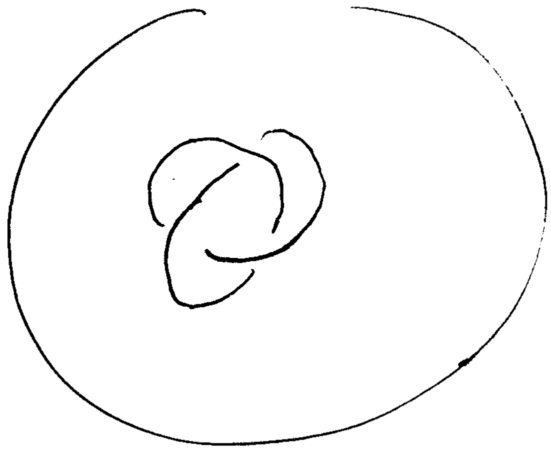
In other words, if one embeds k disjoint knots in S^3 the resulting invariant obtained is simply the product of the invariants ass. to each knot, and one may define

$$\langle \text{KNOT} \rangle = \frac{Z(S^3, \text{KNOT})}{Z(S^3)}$$

with appropriate choice of G, Rep^n.

WE ARE FINALLY IN A POSITION TO NAIL DOWN THE INVARIANT WE OBTAIN.

SUPPOSE WE HAVE A ~~KNOT~~ LINK L EMBEDDED IN S^3:

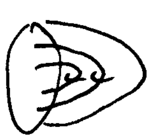


THE ONLY THING STOPPING US UNKNOTTING IT IS THE PRESENCE OF THE CROSSINGS. NEAR A CROSSING WE MAY CUT OUT A LITTLE BALL, AND GET THE PICTURE

M_L

M_R

IN PLACE OF M_R, WE COULD GLUE BACK



X_1

X_2

~~To~~ to get different ~~knots~~ links.

Now in all cases, the bdd is a two-sphere punctured at four points, so if we associate the defining rep of $SL(N)$, and take $k \geq 2$, we get a 2-D vector space associated $\mathcal{H}$ to each. M_L, M_R, X_1, X_2 then give us vectors $\xi_L, \xi_R, \eta_1, \eta_2$ in $\mathcal{H}$, and $Z(L) = \langle \xi_L, \xi_R \rangle$. Gluing in X_1, X_2 instead of L gives us two different knots L_1, L_2, with $Z(L_i) = \langle \xi_L, \eta_i \rangle$. Now, $\mathcal{H}$ is 2-D, so there exists constants $\alpha_i \ni$

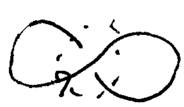
~~$\alpha_0 \langle \xi_L, \xi_R \rangle + \alpha_1 \langle \xi_L, \eta_1 \rangle + \alpha_2 \langle \xi_L, \eta_2 \rangle = 0$~~

$$\alpha_0 \xi_R + \alpha_1 \eta_1 + \alpha_2 \eta_2 = 0.$$

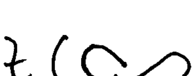
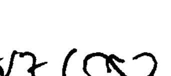
Thus $\alpha_0 Z(L) + \alpha_1 Z(L_1) + \alpha_2 Z(L_2) = 0$, giving us a skein relation, which ~~along with knowing Z(unknot) would~~ uniquel ~~uniquely~~ determines the invariant.

~~It remains to determine the α_i and $Z(O)$.~~

Let us find $Z(O)$. Consider

.

Applying the relation shows

~~Z~~ $\alpha_1 Z($  $) + \alpha_1 Z($  $) + \alpha_2 Z($  $) = 0$, SO THAT

$$z(Q) = -\frac{\alpha_0+\alpha_1}{\alpha_2}.$$

IT REMAINS TO DETERMINE THE CONSTANTS.

IN ORDER TO DO SO WE STUDY THE OPERATOR $B:\mathcal{H}\to\mathcal{H}$ CORRESPONDING TO THE HALF-TWIST.

THEN $\eta_1 = B\eta_R$, $\eta_2 = B^2\eta_R$.

NOW $\mathcal{H}$ IS 2-D, SO B OBEYS THE CHAR. EQN

$$B^2 - \mathrm{Tr}\,B\cdot B + \det B = 0,$$

AND

$$\det B\cdot\eta_R - \mathrm{Tr}\,B\cdot\eta_1 + \eta_2 = 0.$$

IT REMAINS TO DETERMINE THE E-VALUES. MOORE + SEIBERG SHOW

$$\lambda_i = \pm\exp\left(i\pi(2h_R - h_{E_i})\right),$$

WHERE h_R IS A NUMBER ASSOCIATED WITH A REP, THE CONFORMAL WT OF A PRIMARY FIELD, XFMING IN R, AND E_i AS BEFORE. IF R IS THE N-DIM REP^n OF $SU(N)$, THEN

$$\lambda_1 = \exp\left(\frac{i\pi(-N+1)}{N(N+k)}\right),\quad \lambda_2 = -\exp\left(\frac{i\pi(N+1)}{N(N+k)}\right).$$

UP UNTIL NOW WE HAVE BEEN ABLE TO IGNORE FRAMING ISSUES. HOWEVER, B TWISTS UP THE FRAMING, AND IN ORDER TO UNTWIST A HALF TWIST, WE NEED TO MULTIPLY THE CORR. FNS BY $\left(\exp(-2\pi i h_R)\right)^{m}$, WHERE m IS THE # OF HALF TWISTS.

PUTTING THIS ALL TOGETHER, AND DEFINING

$$q = \exp\left(2\pi i / (N+k)\right),$$

WE OBTAIN

$$-q^{N/2} L_+ + \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) L_0 + q^{-N/2} L_- = 0,$$

$$\left(L \to L_+,\ L_1 \to L_0,\ L_2 \to L_-\right)$$

AND

$$\langle \bigcirc \rangle = \frac{q^{N/2} - q^{-N/2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

THE $SU_q(N)$ INVARIANTS.

THE CHERN-SIMONS TQFT MAY BE USED TO STUDY ARBITRARY 3-FOLDS WITH KNOTS (USING SURGERY) BUT TIME UNFORTUNATELY DOES NOT PERMIT US TO EXPLORE THIS.