The Jones Polynomial & Braided Tensor Categories – Paccer Louret ①

Link Finite collection of disjoint (smoothly) embedded circles embedded in $\mathbb{R}^3$

Knot A component of a Link

Link Diagram Projection to $\mathbb{R}^2$ that is at most $2:1$, and intersections transverse, as well as remembering which strand is on top (a generic projection).

Link Number $\ell(k,k')$ two link components; # of overcrossings w/ orientation of k over k':

orientation

$+$ $\quad$ $-$

Framed Link A link along with a section of its normal bundle.

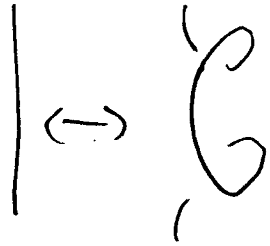
The question:

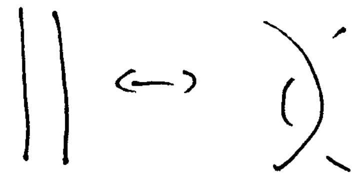
When do two link diagrams come from isotopic links?

The answer:

When they are related by Reidemeister moves & ambient isotopy

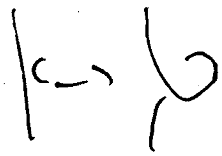
FRAMED F:

Ω_0: 

Ω_1: 

Ω_2: 

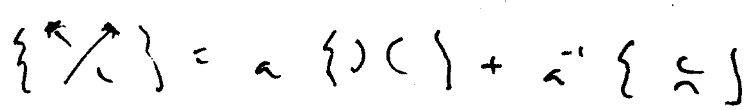
(UNFRAMED) UF

Ω_0':  Ω_1, Ω_2 AS BEFORE

SO, IF YOU CAN GET SOMETHING FROM THE LINK DIAGRAM INVARIANT UNDER REIDEMEISTER MOVES, YOU HAVE A LINK INVARIANT.

JONES POLYNOMIAL

LET $E(a)$ BE A CPLX SPACE GENERATED BY ALL LINK DIAGRAMS MODULO

- AMBIENT ISOTOPY
- $D \cup \{O\} = -(a^2 - a^{-2}) D$
- 

~~(WITH POSITIVE SIGNS DEPENDING ON~~

REG

PACKET (3)

Theorem $D_m \; E(a) = 1$.

[Just resolve link diagram to circles, so everything is generated by the unknot.]

Theorem $\langle\langle L \rangle\rangle$ is invt under Reidemeister moves $\Omega_0, \Omega_1, \Omega_2$. (A framed invt).

Normalise by setting $\langle\langle \emptyset \rangle\rangle = 1$.

Now define $\langle L \rangle = \dfrac{\langle\langle L \rangle\rangle}{-(a^2+a^{-2})}$ ($\langle\;\rangle$ for links, $\langle\langle\;\rangle\rangle$ for diagrams)

$\langle O \rangle = 1$.

In fact $\langle\langle L \rangle\rangle$ is div. by $-(a^2+a^{-2})$, so is a poly in a.

Finally, the Jones polynomial:

For a link L,

$V_L(q) = \langle L \rangle$, with $a = q^{\frac{1}{2}}$, which is a poly in q, q^{-1}; L with 0-framing

Gen framing?

$$V_L(q) = (-1)^{|D|+1} q^{3w(D)/2} \frac{\langle D \rangle (q^{-\frac{1}{2}})}{q+q^{-1}}$$

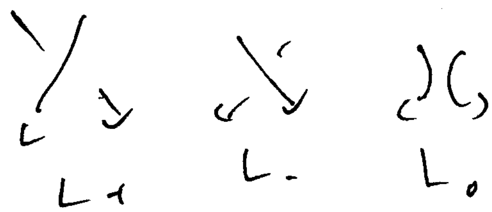
Thm $\exists!$ function V: {nonempty oriented links in $\mathbb{R}^3$} $\to \mathbb{Z}[q, q^{-1}]$

$\ni$

i) If $L \sim L'$ then $V(L) = V(L')$

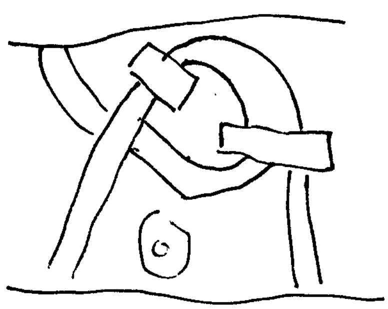
ii) $V(O) = \cancel{(q-q^{-1})}\ 1$

iii) $q^{-2}V(L_+) - q^{2}V(L_-) = (q - q^{-1})V(L_0)$



RIBBON DIAGRAMS:

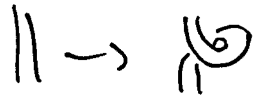
LOOK LIKE



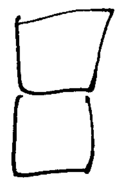
DEF^n IN BK, RTI.

WHAT CAN WE DO WITH THESE DIAGRAMS?

TWISTS:



COMPOSITION:



OTHER STUFF?

LOOK FOR SIMILAR STUFF IN CATEGORIES.

i) MONOIDAL CATEGORIES ...
ii) BRAIDED CATEGORIES
iii) RIGID CATEGORIES
iv) RIBBON CATEGORIES.

i) MONOIDAL CATEGORY GIVES THE IDEA OF A TENSOR PRODUCT.

$$\langle B, \Box, e, \alpha, \beta, \rho \rangle$$

B - CATEGORY

$\Box$ - BIFUNCTOR $B \times B \to B$

$e \in B$, α: NATURAL ISO ("ASSOCIATIVITY")

$$\Box(\Box \times id) \xrightarrow{\sim} \Box(id \times \Box)$$

β: $\Box(e \times id) \underset{\beta}{\cong} 1_B$

ρ: $\Box(id \times e) \underset{\rho}{\cong} 1_B$

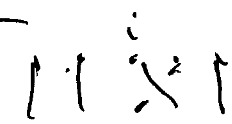
Also $\{\underset{e}{\underset{\parallel}{1}}, V_i, \ldots, V_n\} \subseteq B$

Any ordering of tensoring and insertions of 1 are naturally iso: e.g. $(((V_1 \otimes V_2) \otimes e) \otimes V_3) \underset{!}{\cong} V_1 \otimes (V_2 \otimes V_3)$

ii) <u>Braided</u> ...

<u>Braid group</u> Gen free group on n-generators modulo

$$b_k b_{k+1} b_k = b_{k+1} b_k b_{k+1}$$
$$b_i b_j = b_j b_i \quad \text{if } |i-j| \geq 2.$$

$b_i =$ 

Braided category: want $V_1 \otimes V_2 \cong V_2 \otimes V_1$, but not nec.:

$$\begin{tikzcd} V_1 \otimes V_2 \arrow[r] & V_2 \otimes V_1 \arrow[r] & V_1 \otimes V_1 \\ \arrow[from=1-1, to=1-3, bend right, "\neq id"'] \end{tikzcd}$$

Start w/ a monoidal category

$$\begin{tikzcd} B \times B \arrow[r, "\tau"] & B \times B \arrow[r, "\square"'] \arrow[d, phantom, "\wr\gamma"] & B \\ & B \times B \arrow[r, "\square"] & B \end{tikzcd}$$

γ a natural iso.

γ must play nice with ass. as before.

WHAT YOU WANT IS THAT THE ISO ONLY DEPENDS ON THE ELT IN THE BRAID GROUP TO GET FROM ONE COMP. TO THE OTHER:

$$\begin{tikzcd}
(V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[r] & ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \\
(V_2 \otimes V_1) \otimes (V_3 \otimes V_4) \arrow[d] & ((V_2 \otimes V_1) \otimes V_3) \otimes V_4 \arrow[d] \\
(V_2 \otimes (V_1 \otimes V_3)) \otimes V_4 & (V_2 \otimes (V_1 \otimes V_3)) \otimes V_4 \arrow[l, "="']
\end{tikzcd}$$

iii) $\mathcal{C}$ MONOIDAL CATEGORY, V OBJ IN $\mathcal{C}$.

RIGHT DUAL: V^* & TWO MORPHISMS:

$e_V : V^* \otimes V \to 1$

$i_V : 1 \to V \otimes V^*$

$V \xrightarrow{i \otimes id} V \otimes V^* \otimes V \xrightarrow{id \otimes e_V} V = id_V$

$V^* \xrightarrow{id \otimes i_V} V^* \otimes V \otimes V^* \xrightarrow{e_V \otimes id} V^* = id_V$

$\mathcal{C}$ IS <u>RIGID</u> IF $\exists$ RIGHT AND LEFT DUAL

(LEFT DUAL THE SAME AS EXCEPT ON LEFT :)).

iv) RIBBON CATEGORY: A RIGID BRAIDED TENSOR CATEGORY (8)

w/ FUNCTORIAL ISO $\delta_V : V \xrightarrow{\sim} V^{**}$

w/ PROP $\delta_{V\otimes W} = \delta_V \otimes \delta_W$

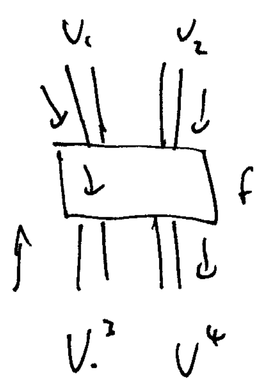
$\delta_1 = id$

$\delta_{V^*} = (\delta_V^*)^{-1}$

CAN NOW DEFINE "TWISTS" OR "BALANCING MORPHISM"

$$\theta_V : V \xrightarrow{\delta_V} V^{**} \xrightarrow{i_V\otimes id} V\otimes V^*\otimes V^{**} \xrightarrow{id\otimes \gamma} V\otimes V^{**}\otimes V^* \xrightarrow{id\otimes e_{V^*}} V.$$

• TO A RIBBON IN A RIBBON GRAPH WE ASSIGN AN ELEMENT OF A RIBBON CATEGORY, AND TO A COUPON WE ASSIGN A MORPHISM AS FOLLOWS:



$\varepsilon_1 = \varepsilon_2 = \varepsilon^4 = +$

$\varepsilon^3 = -$

$(V^i)^+ = V^i$, $(V^i)^- = (V^i)^*$

$f \in Hom(\otimes V_i^{\varepsilon_i}, \otimes (V^j)^{\varepsilon^j})$.

NOW FORM $N(C)$ THE CLASS OF FINITE NUMBER ELTS IN C WITH PARITY (eg, tuples $((V_1,\varepsilon_1)\ldots,(V_n,\varepsilon_n))$), THE OBJECTS.

MORPHISMS, COLOURED RIBBON TANGLES.

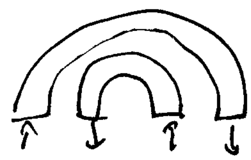
This forms a ribbon category.

$\otimes$ is horizontal composition

Have a braiding action

Dual $\Leftrightarrow$ turning upside down

Unit is $\emptyset$

$\underset{\eta}{\hat{I}\ \check{I}} \otimes \underset{\eta^*}{\hat{I}\ \check{I}} \rightsquigarrow$  morphism $\eta \otimes \eta^* \to id$

other way gives $id \to \eta \otimes \eta^*$.

Th Let $\mathcal{C}$ be a ribbon category. $\exists!$ cov. functor $N(\mathcal{C}) \to \mathcal{C}$ w/ the following prop.

1) Preserves $\otimes$

2)  rainbow $\to e_V$

3)  $\to i_V$

Blah bla ... preserves ribbon structure.

Given an embedded annulus you get a map from $1 \to 1 \in End(V)$, the RT inv of the ribbon.

For you get the Jones poly from sl_q and ...