THE S INVARIANT & THE ~~MINIMAL~~ MILNOR CONJECTURE?

(JAKE RASMUSSEN)

SCOTT II ①

THE S-INVARIANT IS A $\mathbb{R}$-VALUED INVARIANT OF ~~LINKS~~ KNOTS. IT IS THE q-GRADING OF A CERTAIN SPECIAL PIECE OF ~~THE~~ $Kh(K) \supseteq E(K)$.

$\uparrow$ "THE EXCEPTIONAL PAIR".

A CONNECTED COBORDISM FROM $K_1 \to K_2$ IS NON-ZERO WHEN RESTRICTED TO $E(K_1) \to E(K_2)$ ~~...~~.

ASSUMING WE CAN DO THIS, LETS DO AN APPLICATION: THE q-GRADING OF A MAP INDUCED BY A COBORDISM IS ITS EULER CHARACTERISTIC OF THE ASSOCIATED SURFACE. THE SPACE OF POSSIBLE MAPS $E \to E$ IS ALL IN NON-POSITIVE DEGREE. SUPPOSE WE SEE

$$\begin{array}{ccc} q^{s(K_1)} E & \xrightarrow{Kh(\Sigma)|_E} & q^{s(K_2)} E \\ \cap & & \cap \\ Kh(K_1) & \xrightarrow{Kh(\Sigma)} & Kh(K_2) \end{array}$$

SO $\deg(Kh(\Sigma)|_E) = \chi(\Sigma) + s(K_2) - s(K_1) \leq 0$

$$\Rightarrow \chi(\Sigma) \leq s(K_1) - s(K_2)$$

$$\Rightarrow 2 - 2g \leq s(K_1) - s(K_2)$$

$$\Rightarrow g \geq \tfrac{1}{2}(s(K_2) - s(K_1)) + 1$$

BDD ON GENUS OF COB.

SLICE GENUS OF A KNOT $\subset S^3$ IS THE MINIMAL GENUS OF

$$\Sigma \subset B^4, \quad \partial\Sigma = K.$$

MILNOR CONJECTURE IS SOMETHING LIKE

$$\text{slice-genus}(T(p,q)) \overset{?}{=} \tfrac{1}{2}(p-1)(q-1).$$

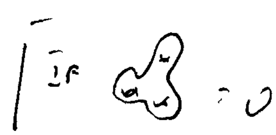
KHOVANOV HOMOLOGY ASSOCIATES TO A 2-SPHERE WITH n MARKED POINTS (AND, HOPEFULLY, A SURFACE WITH n MARKED POINTS) A CATEGORY:

$$W_n = \begin{cases} \text{Ob: tangles in } B^3 \text{ ending in the marked points} \\ \text{Morph: } \mathrm{Hom}(T_1, T_2) = \left\{ \begin{matrix} \text{chain maps up to homotopy} \\ [\![T_1]\!] \to [\![T_2]\!] \end{matrix} \right\} \\ \qquad = Kh(T_1 \cup_{pts} T_2) \end{cases}$$

(INTERPRETED CORRECTLY!!)

FOR W_2 EVERY OBJECT IS ISO TO A DIRECT SUM OF:

$C_0 =$ 



$C_n =$ 

Σ · $^{2} = 0$

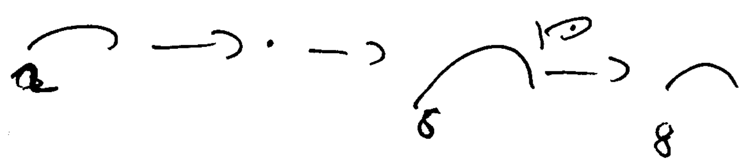
SO $C_{n \geq 2} \cong C_0 \oplus C_0[1]$

AND [ ] AND { } SHIFTS OF THESE.

$\mathcal{C}_n = \begin{cases} \cdot \text{chain complexes, } n \text{ pt diagrams} \\ \cdot \text{chain maps} \end{cases}$

$\mathcal{C}_2^{small} = \begin{cases} \cdot \text{Those chain complexes} \\ \cdot \text{chain maps} \end{cases}$

$\mathcal{C}_2^{\cdot} \cong \mathcal{C}_2^{small}$.

EXAMPLE $[\![$  $]\!] \cong$ 

$= C_0\{8\} \oplus C_1\{6\}[2]$,

PROOF: (SLIGHT GEN. OF GAUSSIAN ELIMINATION) (NO REF.).

PREAMBLE

OBJECTS), $\oplus$

MORPH ARE MATRICES OF POWERS OF)⊙

$\cong$

CATEGORY OF FREE (COMPLEXES) $\mathbb{C}[\alpha]$ MODULES ($\deg \alpha = -2$, all morph degree 0).

SUPPOSE

$$A \to \begin{matrix} B \\ \oplus \\ C \end{matrix} \xrightarrow{\begin{pmatrix} \alpha^n & \alpha^k \\ \alpha^l & \alpha^m \end{pmatrix}} \begin{matrix} D \\ \oplus \\ E \end{matrix} \to F$$

$n \leq k$, $n \leq l$

$\cong$

$$A \to \begin{matrix} B \xrightarrow{\alpha^n} D \\ \oplus \\ C \longrightarrow E \end{matrix} \to F$$

(+ details).

Thus, for tangles:

$$Kh[\![\,\text{tangle}\,]\!] \cong \bigoplus_i C_{n_i}\{k_i\}[l_i]$$

<u>Claim</u> $\exists!$ copy of C_0 (and it's in height 0).

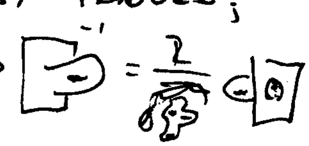
This is the <u>exceptional pair</u>.

For example, the s invariant of the trefoil is ~~...~~ 2.

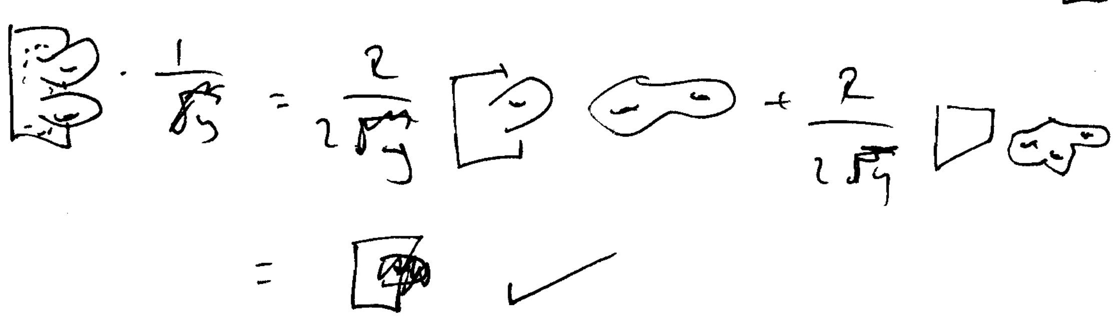
To prove this, we need to learn something about Lee homology:

This is a variation of Kh homology, where the q-grading breaks, and only a filtration survives; it is 2-D (for a knot, for a c-comp link, it is 2^c). It may be obtained from $[\![\;]\!]$ by setting  $= 8$, or simply adjoining  $^{-1/2}$.

 $= y$

Believing these facts, the claim is almost proved; you simply need to show  $^{-1}$ exists ~~is invertible~~ $\Rightarrow$ 

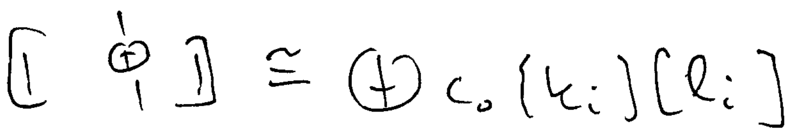
pf

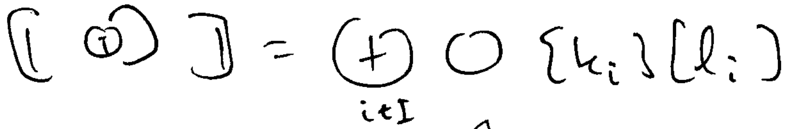


Now C_n 

$\Downarrow ?$

IF y IS INVERTIBLE





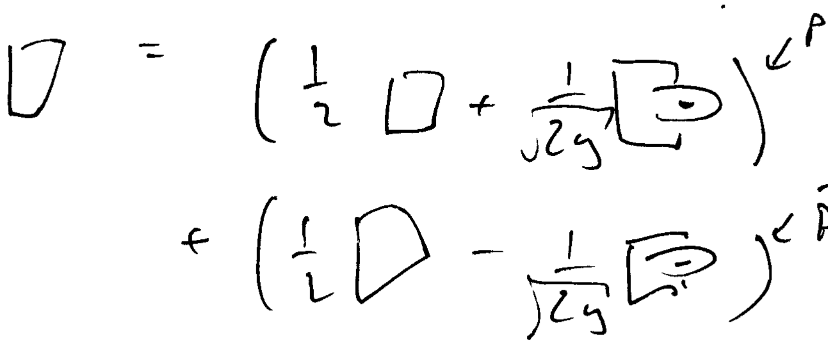
BUT DIM Lie HOM = 2 $\neq 2|I|$ dim

SO $|I| = 1$.

DIDN'T SHOW O IS DEGREE 0!

HOW DO WE SHOW DIM = 0?

IF y^{-1} EXISTS, id ON A STRAND DECOMPOSES INTO TWO ORTHOGONAL IDEMPOTENTS:



CHECK $p^2 = p$: $p^2 =$

$= p$ HOPEFULLY ¿

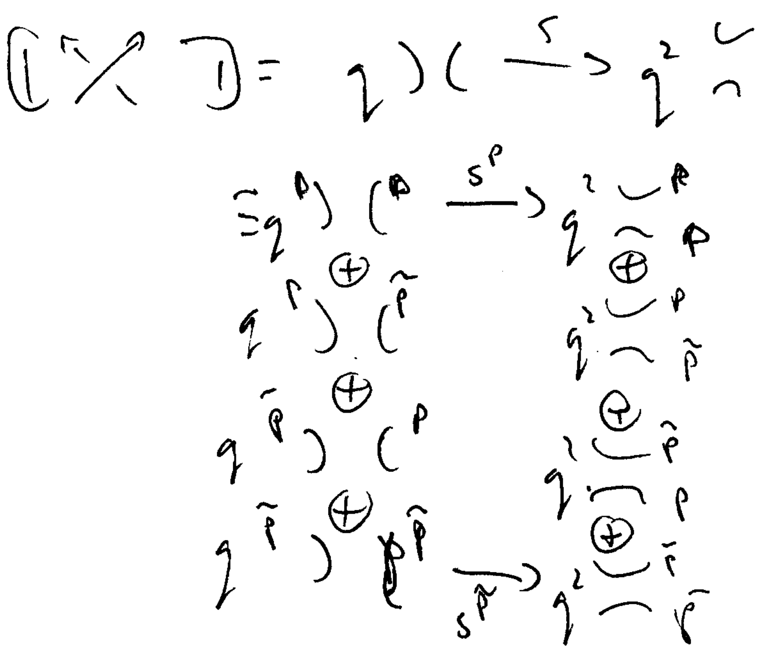
Let's enlarge $Cob(sl_2)$ to $Kar(Cob(sl_2))$, where

$Kar(\mathcal{C}) = \begin{cases} \text{Obj: idempotents } p: O_1 \to O_1 \text{ in } \mathcal{C} \\ \text{Morph } f: p_1 \to p_2 \text{ is a morphism } f: O_1 \to O_2 \\ \quad \text{with } fp_1 = p_2 f. \end{cases}$

So can think of an object as a coloured object in the original category, where the colouring is by the p's.

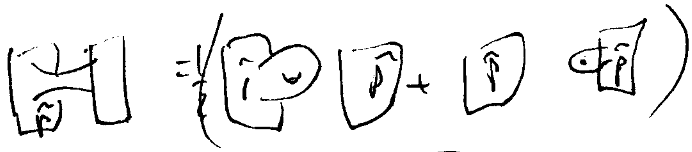
Then $\Big| \cong)^{p} \oplus)^{\tilde{p}}$

So now:

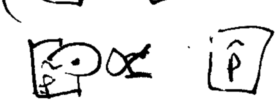


Nearly all maps are zero as red and blue maps collide on saddles.

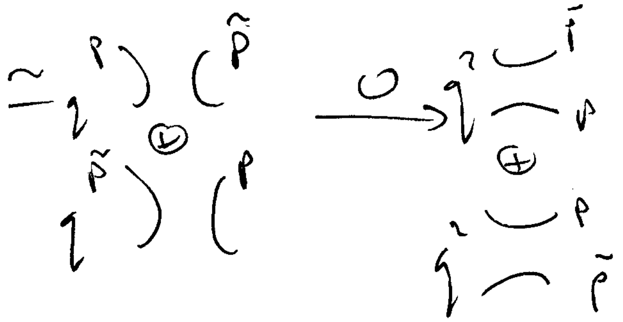
Now $(s^{\tilde{p}})^2 = k1$ as



or M.p. as



SO



SO THE IDEA IS TO COMPUTE THE LIE COH. OF A KNOT BY STICKING THIS IN EVERYWHERE, AND THUS GETTING RID OF ALL DIFFERENTIALS. SO IN ⟦K⟧ THERE ARE NO DIFF, AND THE OBJECTS ARE COLOURED RESOLUTIONS SUBJECT TO THE CONDITION THAT NEAR EVERY CROSSING SITE THE COLOURS SWITCH.

SOME COMBINATORICS SHOW THERE ARE ONLY TWO POSSIBLE COLOURED RESOLUTIONS.

ALTERNATELY COLOURED RESOLUTION

COLOUR SOME INPUT BLUE AND TAKE ALL ORIENTED RESOLUTION.

ORIENTATIONS OF THE KNOT

NEED TO SHOW THE MAP IS NOT ZERO.