JONES POLYNOMIAL EXAMPLES (BOWMAN)

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$$\left.\begin{aligned} &\langle \emptyset \rangle = 1 \\ &\langle \bigcirc D \rangle = (q+q^{-1})\langle D \rangle \\ &\langle \times \rangle = \underset{\text{0-smoothing}}{\langle \asymp \rangle} - q \underset{\text{1-smoothing}}{\langle)(\rangle} \end{aligned}\right\} \text{ KAUFFMAN BRACKET}$$

EG:

$$\langle \infty \rangle = \langle \begin{smallmatrix}\circ\\ \circ\end{smallmatrix} \rangle - q \langle \infty \rangle$$

$$= (q+q^{-1})\langle \bigcirc \rangle - q(q+q^{-1})\langle \emptyset \rangle$$

$$= (q+q^{-1})^2 - q^2 - 1$$

$$= q^{-2} + 1$$

$$J(D) = \langle D \rangle \cdot (-1)^{n_-} q^{n_+ - 2n_-} \qquad \text{(JONES POLYNOMIAL)}$$

n_- = # OF RH CROSSINGS

n_+ = # OF LH CROSSINGS

$$\text{So } J(\infty) = \langle \infty \rangle (-1)^0 q^{1-2\cdot 0} = q + q^{-1}.$$

ON TO KHOVANOV:

THE <u>graded dimension</u> OF A GRADED VECTOR SPACE $V = \bigoplus_m V_m$ IS $q\text{-}\dim V = \sum_m q^m \dim V_m$.

THE <u>shift operator</u> ~~IS~~ ON A GR. V.S. IS

$$[V\{\ell\}]_{(m)} = [V]_{m-\ell}.$$

THEN $q\text{-}\dim V\{\ell\} = q^\ell \, q\text{-}\dim V$.

So let

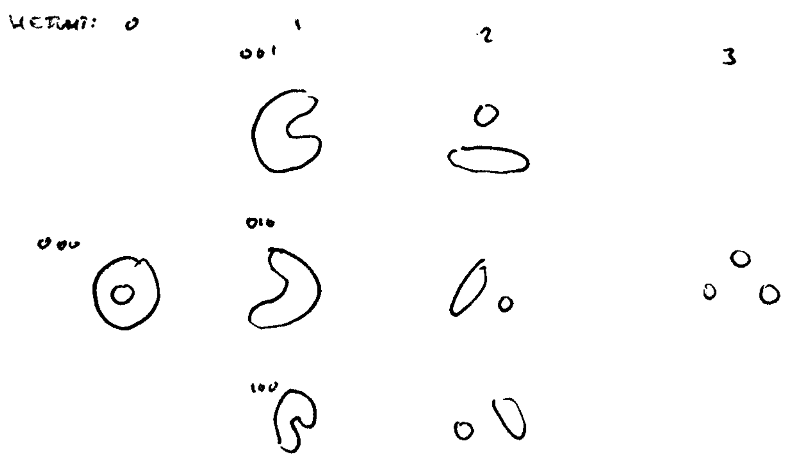
$V = \text{Span}\{v_+, v_-\}$, $\deg v_+ = 1$, $\deg v_- = -1$,
$\Rightarrow q\text{-dim}\, V = q + q^{-1}$.

THE <u>HEIGHT</u> OF A SMOOTHING OF A LINK DIAGRAM IS THE # OF 2 SMOOTHINGS.

<u>Example</u>

$n_+ = 3$
$n_- = 3$

SMOOTHINGS:



NOW ASSIGN V TO EACH UNKNOT, AND SHIFT DEGREE BY HEIGHT:

$$\to V^{\otimes 2}\{0\} \to \bigoplus_{i=1}^{3} V\{1\} \to \bigoplus_{i=1}^{3} V^{\otimes 2}\{2\} \to V^{\otimes 3} \to$$

FOR NOW, IMAGINE $\exists$ MAPS MAKING THIS A CHAIN COMPLEX.

You could then calculate the Euler characteristic of the "complex", (SB) ③

namely

$$\chi(\mathcal{C}) = \sum (-1)^{deg}\, q\text{-}\dim V^{deg}$$

deg shift

$$= (q+q^{-1})^{\otimes 2} - 3q(q+q^{-1}) + 3q^2(q+q^{-1})^2 - q^3(q+q^{-1})^3$$

which miraculously is ~~K~~ $\langle$ $\rangle$

Recall $J(D) = \langle D\rangle (-1)^{n_-} q^{n_+ - 2n_-}$

So define $\mathcal{C}' = \mathcal{C}\{n_+ - 2n_-\}[n_-]$ Then $J(\;) = \chi(\mathcal{C}')$

where this shifts the deg of the complex.

We want now to define maps.

For example, we want a map from

$V\otimes V \to V$)

or, say

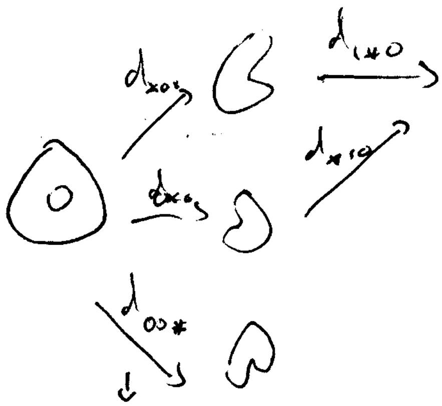
$V \to V\otimes V$

Define $m: V\otimes V \to V$ by $v_+\otimes v_- \mapsto v_-$ $\quad v_+\otimes v_+ \mapsto v_+$
$v_-\otimes v_+ \mapsto v_-$ $\quad v_-\otimes v_- \mapsto 0$

Def $\Delta: V \to V\otimes V$ by $v_+ \mapsto v_+\otimes v_- + v_-\otimes v_+$
$v_- \mapsto v_-\otimes v_-$

So, whenever you go from two circles to one, you use m, and from one to two, you need Δ. But, you need signs.

So, going back to trefoil:



sort of

$d_0 = \cancel{\Sigma}\, d_{*00} \oplus d_{0*0} \oplus d_{00*} = \begin{pmatrix} d_{*00} \\ d_{0*0} \\ d_{00*} \end{pmatrix}$

d_{ijk} are Δ or m,

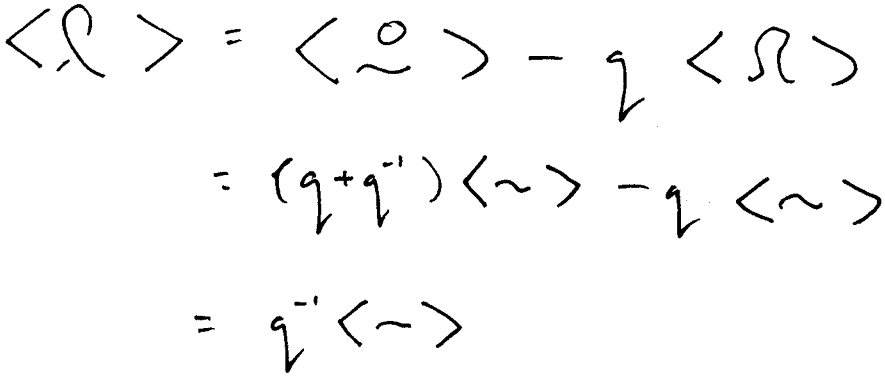
Sign is $\cancel{\#\Delta}(-1)^{\#1 \text{ before } *}$.

Then every square anti-commutes (as m, Δ commute), so $d^2 = 0$ at the bottom. Also, $\deg d$ is $O(?)$, so one has a complex.

The Khovanov homology is the homology of this complex.

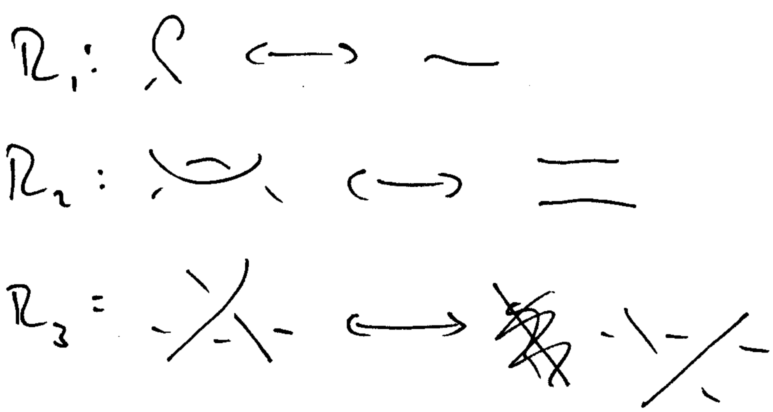
ONTO JONATHAN WILLIAMS:

PROVING THE ~~GRADED~~ HOMOLOGY IS INVT OF KNOTS, SO INVARIANCE UNDER REIDEMEISTER MOVES:
FOR THE JONES POLY, FOR EXAMPLE



WHICH SEEMS ODD, BUT RECALL YOU HAVE TO SHIFT BY CROSSINGS.

THE REIDEMEISTER MOVES:



TWO LEMMAS:

1) Let $C' \subset C$. If $H(C') = 0$, then $H(C/C') = H(C)$.

2) If $H(C/C')$, then $H(C) = H(C')$.

So now, to every link L diagram you get a cube of smoothings, and you form the chain complex as before. The Khovanov bracket $\llbracket L \rrbracket$ of a link is this complex.

Now we want to show invariance under Reidemeister:

$$R_1 : \llbracket \ell \rrbracket = \left(\llbracket \overset{\circ}{\sim} \rrbracket \xrightarrow{m} \llbracket \Omega \rrbracket \{1\} \right) = C$$

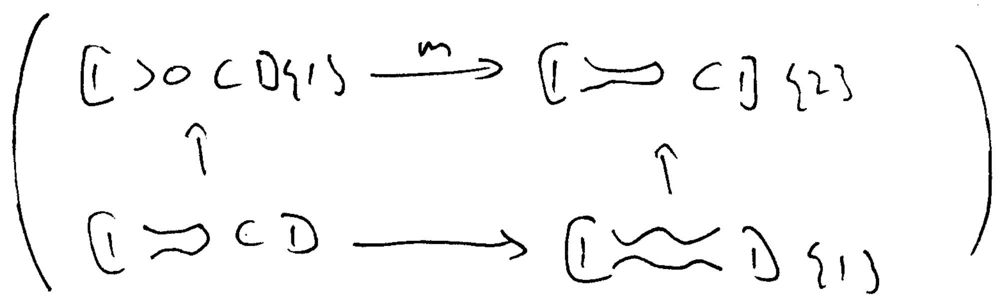
Define $C' = \left(\llbracket \overset{\circ}{\sim} \rrbracket_{v_+} \xrightarrow{m} \llbracket \Omega \rrbracket \{1\} \right)$ (which is a subcomplex, as weight increases)

Now v_+ is a unit for m, so the complex is acyclic.

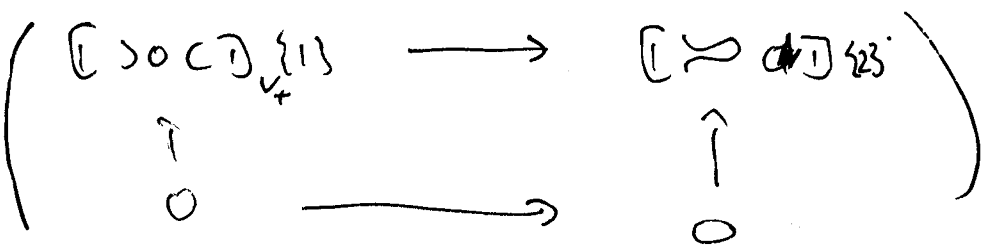
Thus $H(C/C') = H(C)$.

Now
$$C/C' = \left(\llbracket \overset{\circ}{\sim} \rrbracket \Big|_{v_+ = 0} \xrightarrow{m} 0 \right)$$
$$\simeq \left(\llbracket \sim \rrbracket \big|_{v_-} \xrightarrow{m} 0 \right)$$
$$\simeq \llbracket \sim \rrbracket \{-1\}.$$

R2: 

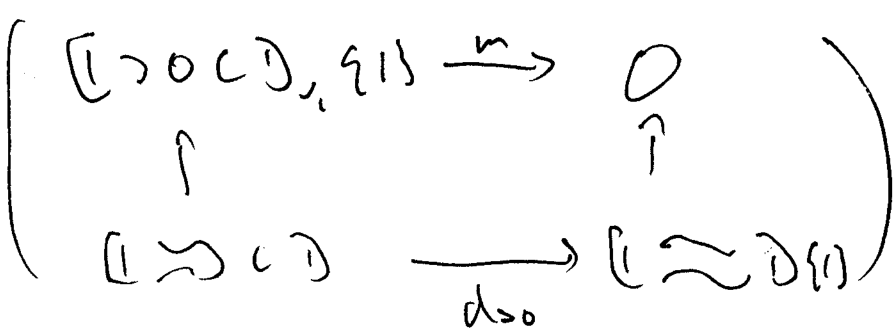
$=$  C^*

Look at C': (a subcomplex for weight reasons).



Claim, C' is acyclic, as m is an iso when restricted to v_+.

So now get C/C'



$\alpha : [DLD] \xrightarrow{\Delta \oplus d_{so}} [TLD]_{v_-}\{1\} \oplus [BRD]\{1\} \to 0$

Write

$\alpha \to (\beta, \tau\beta) \to 0$ $\qquad \tau = \pm d_{so}\Delta^{-1}$ [Δ is iso].

So the subcomplex is

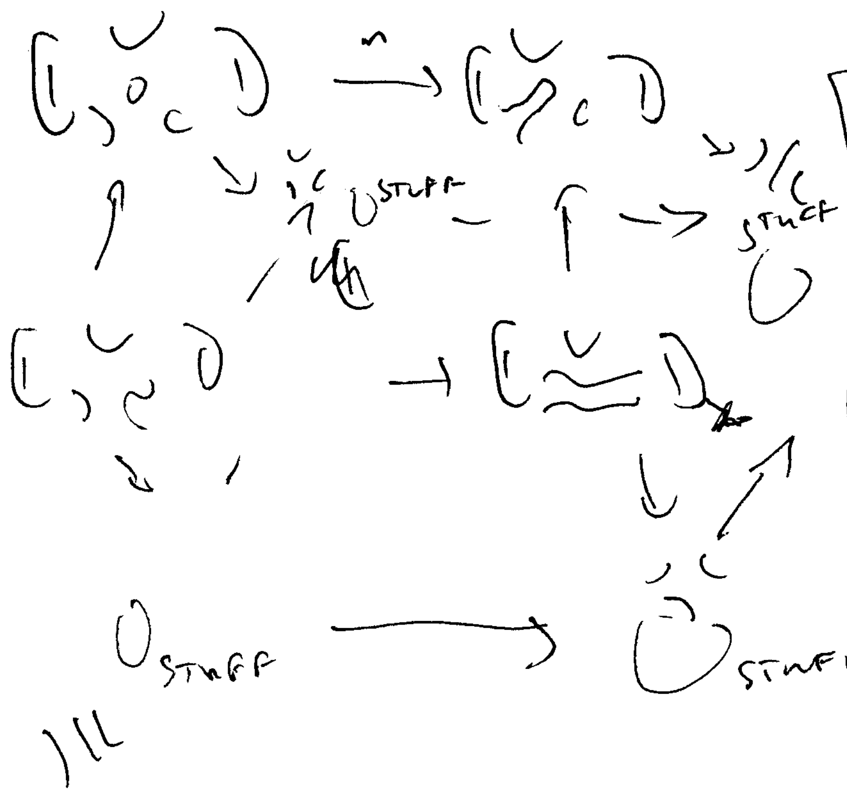
$$C'' \llbracket BCD \rrbracket \to g\tau \to 0$$, I think.

Then $(C/C')/C''$ is just one $\llbracket BRD \rrbracket$

$$H\left((C/C')/C''\right) = H\llbracket \asymp \rrbracket\{1\} = H\llbracket C \rrbracket$$

$\square$

R3: 3D cube



I give up...

Look at Barnatan for picture

BL's corr. to

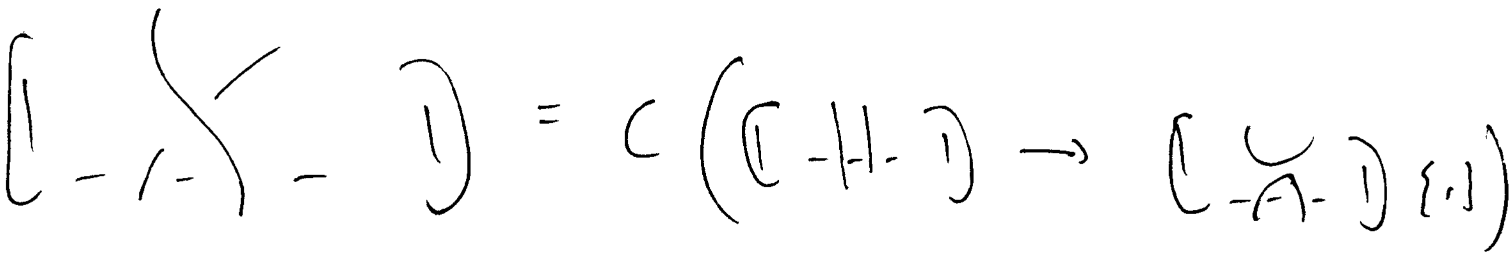
so are auto. iso.

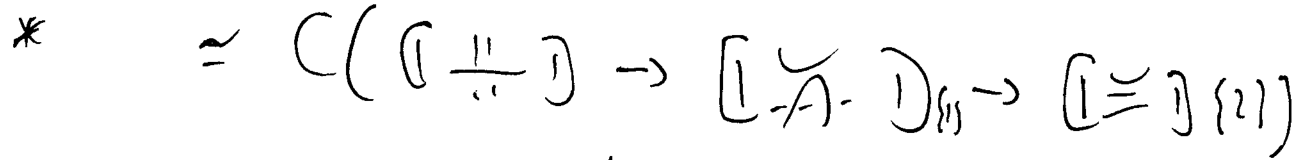
Top.

R3 works thanks to R1 R2.

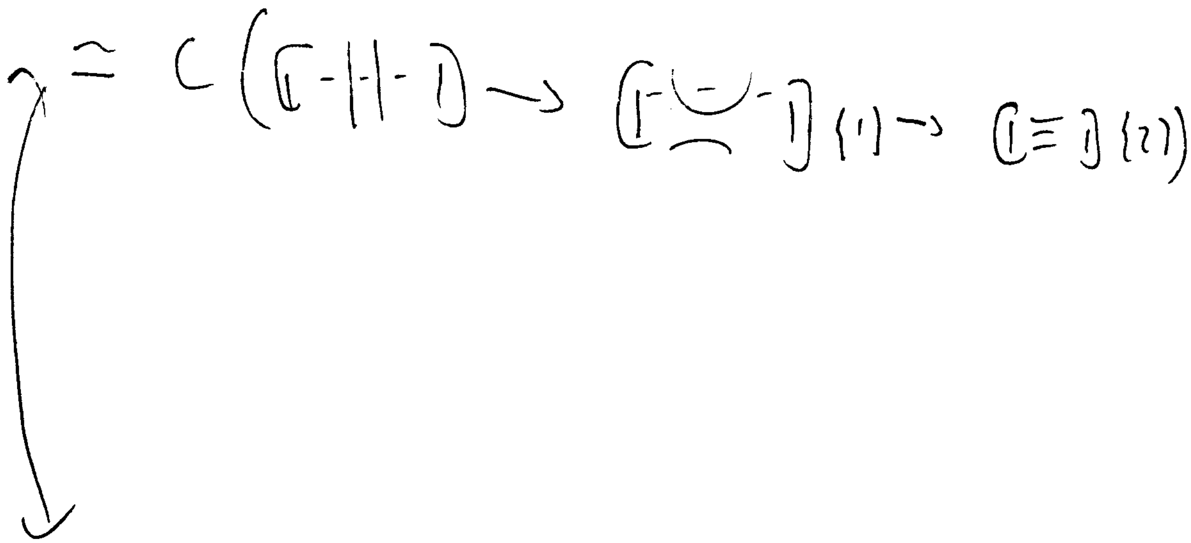
Appendix A slicker proof of R3.

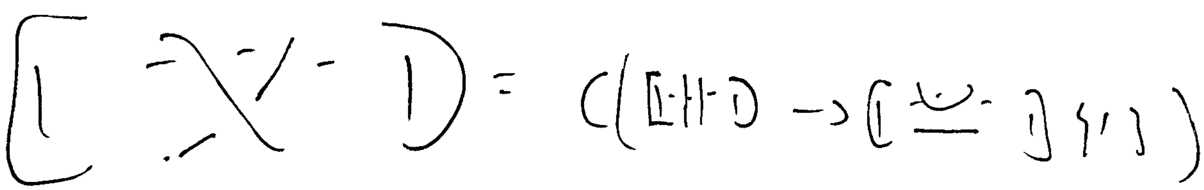
IDEA

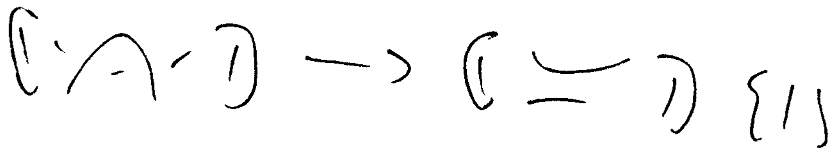




!! by calc.





*R2: 

THIS IS A STRONG DEF. RETRACT.

$$C(f) \underset{\text{htpy. eq.}}{\simeq} C(rf)$$

strong ~~of~~ def retr.