

## Problem Set # 1

M382E: Algebraic Topology

Due: September 9, 2008

Homeworks will be posted on Tuesday. They are to hand in. I urge you to work in groups and to come to office hours to discuss the problems and the class. But write up the homeworks individually. Do it nicely and neatly—write good mathematics! Do not hand in scratch work.

Many problems are hard. Do not get discouraged. When you get stuck seek help of all sorts. Don't worry if you don't hand in solutions to all problems. Do the problems you hand in well.

You will need your class notes to make sense of some of the problems.

1. Check carefully that the equivalence relation used in the definition of the bordism theory  $MO_n$  is in fact an equivalence relation.
2. The definition of  $MSO_n$  relies on the notion of an *orientation* of a smooth manifold. We review some basics here.
  - (a) Let  $V$  be a finite dimensional real vector space. Let  $\mathcal{B}(V)$  denote the set of (ordered) bases of  $V$ . If  $\dim V = n$ , then a basis is an isomorphism  $\mathbb{R}^n \rightarrow V$ . Two bases are related by a unique isomorphism  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ , which is represented by an invertible  $n \times n$  matrix. Verify these facts.
  - (b) Define an equivalence relation on  $\mathcal{B}(V)$  by declaring two bases to be equivalent if the matrix which relates them has positive determinant. Check that this is an equivalence relation and there are two equivalence classes.
  - (c) Let  $M$  be a smooth finite dimensional manifold. Use the bases on each tangent space and the equivalence relation in part (b) to construct the *orientation double cover*  $\hat{M} \rightarrow M$ .
  - (d) An *orientation* of  $M$  is defined to be a section of  $\hat{M} \rightarrow M$ . Show that a connected orientable manifold has two orientations. Show that  $\mathbb{R}P^2$  is not orientable.
  - (e) If  $M$  is a manifold with boundary, then an orientation of  $M$  induces an orientation of the boundary  $\partial M$  using ONF, where ONF="one never forgets"="outward normal first". Explain.
3. Some basics about categories. This is for now a useful framework in which to state some properties of homology theories. A *category*  $\mathcal{C}$  consists of a collection  $\text{ob } \mathcal{C}$  of *objects*; for each pair  $a, b \in \text{ob } \mathcal{C}$  a set  $\mathcal{C}(a, b)$  of *morphisms* from  $a$  to  $b$ ; a distinguished identity morphism in  $\mathcal{C}(a, a)$  for all  $a \in \text{ob } \mathcal{C}$ ; and an associative composition law  $\mathcal{C}(a, b) \times \mathcal{C}(b, c) \rightarrow \mathcal{C}(a, c)$ . (The totality of objects may not be a set—it is in a *small* category—but we will not worry about foundational set theory.) The objects of a category are not necessarily sets and morphisms are not necessarily maps of sets.

- (a) Construct a category  $Top$  whose objects are topological spaces and whose morphisms are continuous maps.
- (b) Construct a category  $Ab$  whose objects are abelian groups and whose morphisms are group homomorphisms.
- (c) Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. Using your algebraic sensibilities define the correct notion of a map  $\mathcal{C} \rightarrow \mathcal{D}$ . This is called a *functor*. Verify that our bordism constructions  $MO_n$ ,  $MSO_n$ ,  $\widetilde{MO}_n$ ,  $\widetilde{MSO}_n$  are functors  $Top \rightarrow Ab$ .
- (d) Let  $G$  be a discrete group. Construct from  $G$  a category with one object.
4. Compute  $\widetilde{MSO}_1(\mathbb{RP}^2)$ . You may want to use the Mayer-Vietoris sequence: recall that  $\mathbb{RP}^2$  is the union of a 2-disk and a Möbius band. Use the homotopy invariance of  $\widetilde{MSO}$ .
5. Recall we define the degree of a continuous map  $f: S^n \rightarrow S^n$  to be the induced map on  $\widetilde{MSO}_n(S^n)$ .
- (a) Compute the degree of a reflection. Of the antipodal map (which is a composition of reflections).
- (b) Show that  $S^n$  has a nonvanishing vector field if and only if  $n$  is odd.
6. Prove by induction  $\widetilde{MSO}_q(S^n) = 0$  for  $q < n$ . You will need the computation of  $MSO_0(X)$  for a space  $X$ .
7. Verify that  $MO_2(\text{point}) \neq 0$  by proving that  $\mathbb{RP}^2$  does not bound a compact 3-manifold. Here is one method. Recall that a finite dimensional real vector space  $V$  has associated vector spaces  $\bigwedge^q V$ , its exterior powers, and that the top exterior power  $\text{Det } V$  is one-dimensional. (Check that the two components of  $\text{Det } V \setminus \{0\}$  may be identified with the two equivalence classes in 2(b).) Apply to the tangent bundle of any manifold  $M$  to construct a real line bundle  $\text{Det } M \rightarrow M$ . For  $M = \mathbb{RP}^2$  construct a section of  $\text{Det } \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  which has zeros along some  $\mathbb{RP}^1 \subset \mathbb{RP}^2$ . (In the model of  $\mathbb{RP}^2$  which is the 2-disk with antipodal points on the boundary identified, an  $\mathbb{RP}^1 \subset \mathbb{RP}^2$  is a diameter of the disk; in the model of lines in  $\mathbb{R}^3$  it is the subspace of lines contained in a 2-plane in  $\mathbb{R}^3$ .) Fix two such sections whose zero sets are transverse. Verify that the mod 2 intersection number of these zero sets is nonzero. Now suppose  $\mathbb{RP}^2 = \partial M$  for some compact 3-manifold  $M$ . Verify that  $\text{Det } M \rightarrow M$  restricts on the boundary to  $\text{Det } \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$ . Use the transversality theory in Guillemin-Pollack to extend each of the two sections over  $\mathbb{RP}^2$  to sections of  $\text{Det } M \rightarrow M$  which are transverse to the zero section and transverse to each other. What can you conclude about the zero sets? What do you know from intersection theory? Do you see a contradiction?
8. Tell what you learn from the statement that each of the following is an exact sequence of abelian groups. Verify your statements carefully from the definition.
- (a)  $0 \rightarrow A \xrightarrow{f} B$

$$(b) A \xrightarrow{f} B \rightarrow 0$$

$$(c) 0 \rightarrow A \xrightarrow{f} B \rightarrow 0$$

$$(d) 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$