Problem Set # 10

M382E: Algebraic Topology

Due: November 11, 2008

I recommend reading material in Hatcher we will not cover in lecture, for example some of the appendices in Chapter 3.

Problems in Hatcher

Section 3.2 (page 228): 10, 16, 18
Section 3.A (page 267): 1, 2
Section 3.C (page 291): 4, 5

Other Problems

1. Let G be a finite group. It is a topological group when endowed with the discrete topology.
   (a) According to our discussion in lecture, the cohomology $H^\bullet(G; R)$ is a Hopf algebra for any commutative ring R. Write a basis and write the coproduct in that basis.
   (b) The homology $H_\bullet(G; R)$ is also a Hopf algebra. The coproduct is obtained from the map induced on homology by the diagonal inclusion $\Delta\colon G \hookrightarrow G \times G$. The product is the map induced on homology by the product $\mu\colon G \times G \to G$. Can you recognize this Hopf algebra?
   (c) An element a in a Hopf algebra A is *grouplike* if $\Delta(a) = a \otimes a$. Are there grouplike elements in the Hopf algebras of (a) or (b)?
   (d) Let R be a field of characteristic zero. Are the Hopf algebras in (a) and (b) exterior algebras? If not, why does this not violate the Hopf theorem proved in lecture?

2. For each part of this problem construct a space X whose cohomology ring is as indicated or prove that none exists.
   (a) $H^\bullet(X; \mathbb{F}_3) \cong \mathbb{F}_3[x]/(x^2)$, where $\deg x = 5$.
   (b) $H^\bullet(X; \mathbb{F}_2) \cong \mathbb{F}_2[x, y]/(x^2)$, where $\deg x = 5$, $\deg y = 1$.
   (c) $H^\bullet(X; \mathbb{Z}) \cong \mathbb{Z}[x, y, z]/(2x, z^4)$, where $\deg x = 1$, $\deg y = 2$, $\deg z = 2$.
   (d) $H^\bullet(X; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^3)$, where $\deg x = 2$.