

Problem Set # 11

M382E: Algebraic Topology

Due: November 18, 2008

Problems in Hatcher

Section 3.3 (page 257): 3, 6, 7, 8, 10, 11, 17, 20, 21, 24, 26

Other Problems

1. Let X be a space and R a ring. Recall that the cap product is defined by the adjointness equation

$$\langle \alpha, \beta \frown a \rangle = \langle \alpha \smile \beta, a \rangle$$

for $\alpha \in C^q(X; R)$, $\beta \in C^r(X; R)$, $a \in C_{q+r}(X; R)$. Prove the following identities.

- (a) \frown is bilinear
 - (b) $1 \frown a = a$
 - (c) $(\beta \smile \beta') \frown a = \beta \frown (\beta' \frown a)$
 - (d) $\partial(\beta \frown a) = \delta\beta \frown a + (-1)^{|\beta|}\beta \frown \partial a$
 - (e) Verify that these relations show that homology is a graded module over cohomology.
2. Suppose a compact topological n -manifold M has a finite CW structure \mathcal{T} , which leads to a chain complex

$$0 \longleftarrow C_0 \longleftarrow C_1 \longleftarrow \cdots \longleftarrow C_n \longleftarrow 0$$

of finitely generated free abelian groups.

- (a) A dual cell structure, if it exists, has a 0-simplex for every n -simplex of \mathcal{T} , a 1-simplex for every $(n-1)$ -simplex of \mathcal{T} , etc. What is its chain complex? What would be the nicest answer for the differential?
 - (b) Carry this out for $M = S^1$. For M a 2-dimensional torus.
 - (c) Given these dual cell structures, prove Poincaré duality.
3. Let \mathbb{A}^n be affine space with fixed origin 0. The group of affine transformations which fixes 0 is $GL_n(\mathbb{R})$. Compute its action on $H_n(\mathbb{A}^n, \mathbb{A}^n - \{0\})$.

4. Let X be a space, A, B open subspaces, and R a ring.

(a) Show that the cup product determines a map

$$\smile: H^q(X, A; R) \times H^r(X, B; R) \longrightarrow H^{q+r}(X, A \cup B; R)$$

Does this product also exist on the corresponding cochains?

(b) Show that there is an induced cap product

$$\frown: H^q(X, A; R) \times H_{q+r}(X, A \cup B; R) \longrightarrow H_r(X, B; R)$$

We used the case $A = \emptyset$ in lecture.

5. Compute the colimit (direct limit) of

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{5} \mathbb{Z} \xrightarrow{7} \mathbb{Z} \longrightarrow \dots$$