Problem Set # 3

M382E: Algebraic Topology

Due: September 23, 2008

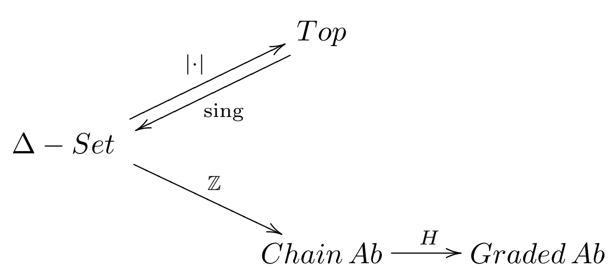
There were errors in Problem Set #2 which are now corrected in the new online version of the problem set. (Please email me with errors you discover in the future.)

Problems in Hatcher

Section 2.1 (page 131): 11, 12, 16, 21, 27, 29

Other Problems

1. Recall our basic diagram



Define each of the four entries as a category—specify the morphisms—and show that each of the four arrows is a functor.

2. (a) Complete the proof of the Snake Lemma.

(b) Prove the following stronger version of the 5-lemma. In the commutative diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E' \end{array}$$

suppose that the rows are exact. If β and δ are surjective and ϵ is injective, prove that γ is surjective. If β and δ are injective and α is surjective, prove that γ is injective.

3. (a) Suppose $(C'_\bullet, \partial')$ and $(C''_\bullet, \partial'')$ are chain complexes supported in nonnegative degrees: $C'_n = C''_n = 0$ if $n < 0$. Define the tensor product complex $(C_\bullet, \partial)$ by

$$C_n = \bigoplus_{p+q=n} C'_p \otimes C''_q$$

and

$$\partial(c'_p \otimes c''_q) = \partial' c'_p \otimes c''_q + (-1)^p c'_p \otimes \partial'' c''_q$$

Verify that $(C_\bullet, \partial)$ is a chain complex. Show that we can replace the support condition by the condition that one of the complexes be finitely supported. Notice the use of the *Koszul sign rule* which should be familiar from the rule for the exterior derivative d of the wedge product of differential forms.

(b) Consider the Δ-set with two 0-simplices p, q and a single 1-simplex a with $d_0(a) = q$ and $d_1(a) = p$. Show that its geometric realization is homeomorphic to the interval $I = [0, 1]$ and that its chain complex has the shape

$$\mathbb{Z} \oplus \mathbb{Z} \longleftarrow \mathbb{Z}$$

What is the boundary map? Call this chain complex $(C'_\bullet, \partial')$.

(c) Let $(C''_\bullet, \partial'')$ and $(D_\bullet, \partial)$ be arbitrary chain complexes. What data defines a chain map $C' \otimes C'' \to D$? Express your answer in terms of C'' and D only. Do you see anything familiar?

4. Let V be a finite dimensional real inner product space. Define $St_k(V)$ to be the set of k-tuples of orthonormal vectors in V. In other words, $St_k(V)$ is the set of isometries $\mathbb{R}^k \to V$ where $\mathbb{R}^k$ has its standard inner product. Note that necessarily $k \leq \dim V$.

(a) Construct a free right action of the orthogonal group $O(k)$ on $St_k(V)$. The quotient is denoted $Gr_k(V)$. How can you describe it? What is it in case $k = 1$? What is $St_1(V)$?

(b) Topologize $St_k(V)$ and $Gr_k(V)$. Can you show that they are smooth manifolds? Or at least that they are *topological manifolds*, i.e., spaces which are *locally Euclidean* in the sense that each point has a neighborhood homeomorphic to an open set in an affine space. What are their dimensions?

(c) What familiar space is $St_2(\mathbb{R}^3)$?

(d) Construct a continuous map $f\colon St_k(V) \to St_{k-1}(V)$ by fixing an isometry $\mathbb{R}^{k-1} \hookrightarrow \mathbb{R}^k$. What is the image of f? What is a typical fiber of f?

(e) Are there complex and quaternionic analogs?