# Problem Set # 4

M382E: Algebraic Topology

Due: October 1, 2008

Class is cancelled September 30 and October 9. Make-up lectures will be held in RLM 9.166 at 5:00 on October 1 and October 15.

### Problems in Hatcher

Section 2.1 (page 131): 17, 18
Section 2.2 (page 155): 2, 7, 28, 29

### Other Problems

1. Let $Y$ be a compact metric space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ a collection of subsets of $Y$ whose interiors cover $Y$. Prove there exists $\epsilon > 0$ such that any ball in $Y$ of radius less than $\epsilon$ is contained in some $U_\alpha$.

2. (a) Construct a $\Delta$-set whose geometric realization is the standard $n$-simplex $\Delta^n$. It should have $\binom{n+1}{q+1}$ $q$-simplices.

   (b) In terms of your answer construct a $\Delta$-set which represents the (first) barycentric subdivision of $\Delta^n$. What is a $q$-simplex of the barycentric subdivision in terms of the simplices of the original?

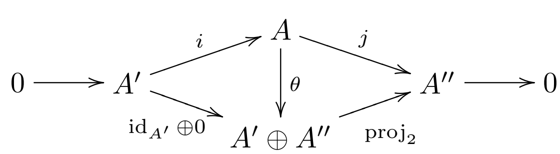
   (c) Use your answer in (b) to construct an abstract barycentric subdivision of any $\Delta$-set. The answer is a new $\Delta$-set.

3. (a) Show that for the short exact sequence

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$$

   the following are equivalent:

   (i) There exists a homomorphism $p\colon A \to A'$ such that $p \circ i = \mathrm{id}_{A'}$
   (ii) There exists a homomorphism $s\colon A'' \to A$ such that $j \circ s = \mathrm{id}_{A''}$
   (iii) There exists an isomorphism $\theta\colon A \to A' \oplus A''$ such that the diagram



commutes. In case any, hence all, hold we say the exact sequence is *split* and the given data provides a *splitting*.

(b) Suppose

$$0 \longrightarrow C'_\bullet \longrightarrow C_\bullet \rightleftarrows C''_\bullet \longrightarrow 0$$

is a split short exact sequence of chain complexes. Show that the boundary map in the long exact sequence of homology groups vanishes. Conclude that this sequence decomposes into short exact sequences, one for each integer.

4. Derive the Mayer-Vietoris sequence for reduced homology from the Mayer-Vietoris sequence for unreduced homology.

5. I have used affine spaces freely in lecture. Just in case you are not familiar with them, I provide this exercise. An *affine space* $A$ is a set together with an associated vector space $V$ of translations and a simply transitive action of $V$ on $A$. The result of translation by a vector $\xi \in V$ on a point $p \in A$ is written as $p + \xi \in A$.

(a) Define the notion of an affine map $f\colon A \to B$ between affine spaces. Let $\mathrm{Aff}(A, B)$ denote the set of affine maps. Then show the subset $\mathrm{Aff}'(A) \subset \mathrm{Aff}(A, A)$ of invertible affine maps is a group under composition and it fits into the short exact sequence

$$1 \longrightarrow V \longrightarrow \mathrm{Aff}'(A) \longrightarrow \mathrm{Aut}(V) \longrightarrow 1$$

of (nonabelian) groups, where $\mathrm{Aut}(V)$ is the group of linear automorphisms of the vector space $V$. Show that each point of $A$ determines a splitting of this sequence.

(b) Define the notion of an affine subspace of $A$. Show that an invertible map $f\colon A \to B$ is affine if and only if it maps affine lines in $A$ to affine lines in $B$.

(c) If $F \subset A$ is a finite set, define its center of mass. Prove that if $f\colon A \to B$ is affine, then the center of mass of $f(F)$ is the image under $f$ of the center of mass of $F$.

6. Compute $H_\bullet(\mathbb{CP}^n)$.

7. (a) Show that $(\Delta^n, \partial\Delta^n)$ is a good pair.

(b) Show that $(\partial\Delta^n, \Lambda)$ is a good pair, where $\Lambda$ is the union of all but one face in $\partial\Delta^n$.

(c) Let $S$ be a $\Delta$-set and $X_0 \subset X_1 \subset X_2 \subset \cdots$ the filtration of its geometric realization $X = |S|$. Show that $(X_p, X_{p-1})$ is a good pair.