

## Problem Set # 5

M382E: Algebraic Topology

Due: October 7, 2008

Class is cancelled October 9. The make-up lecture will be held in RLM 9.166 at 5:00 on October 15.

### Problems in Hatcher

Section 2.2 (page 155): 9, 12, 13, 15, 18, 19, 32

### Other Problems

1. Let  $X$  be a space and  $f_t: S^{n-1} \rightarrow X$ ,  $0 \leq t \leq 1$  a homotopy. Prove that the spaces  $X_{f_t} = X \cup_{f_t} D^n$  obtained by attaching an  $n$ -cell via  $f_t$  are all homotopy equivalent.
2. Recall the importance of *goodness* when it comes to pairs: the relative homology is then the reduced homology of the quotient space.
  - (a) Let  $X$  be a space,  $f: S^{n-1} \rightarrow X$  a continuous map, and  $X_f = X \cup_f D^n$  the space obtained by attaching an  $n$ -cell via  $f$ . Show that  $(X_f, X)$  is a good pair.
  - (b) Let  $X$  be a CW complex and  $A \subset X$  a subcomplex. Show that  $(X, A)$  is a good pair.
3.
  - (a) Construct a cell decomposition of  $\mathbb{C}P^n$  with  $n + 1$  cells each of even dimension. What are the attaching maps? What is the cellular chain complex? Why is the homology so easy to compute? (Hint: You may want to proceed inductively by identifying first the top cell. For that fix an  $n$  dimensional subspace of  $\mathbb{C}^{n+1}$  and consider the lines which are not contained in it. Identify them as graphs.)
  - (b) The quaternionic projective space  $\mathbb{H}P^n$  is the space of quaternionic lines (one-dimensional subspaces) of  $\mathbb{H}P^{n+1}$ . (It may bother you that the quaternions  $\mathbb{H}$  are not a field. Should it?) Can you see that it is a manifold? Of what dimension? Construct a cell decomposition. What are the attaching maps? What is the homology of  $\mathbb{H}P^n$ ? Can you recognize  $\mathbb{H}P^1$ ?
4. Construct a cell decomposition of the Grassmannian  $Gr_2(\mathbb{R}^4)$  of 2-planes in real 4-space. Can you compute the homology?
5. Let  $X$  be a CW complex and  $C \subset X$  a compact subset.
  - (a) Prove that  $C$  meets only finitely many cells of  $X$ .
  - (b) Prove that  $C$  is contained in a finite subcomplex of  $X$ .
6. Let  $S$  be a  $\Delta$ -set. Construct a natural CW structure on  $|S|$ .