Problem Set # 6

M382E: Algebraic Topology

Due: October 14, 2008

Class is cancelled October 9. The make-up lecture will be held in RLM 9.166 at 5:00 on October 15.

Problems in Hatcher

Section 2.2 (page 155): 2, 8, 10, 17, 26, 34

Other Problems

1. (a) Let X be a CW complex, G a finite group which acts freely on X in such a way that it permutes the cells. Thus if A^n is the set of n-cells, the free action of G on the space X induces a free action of G on the set A^n. Construct a CW structure on X/G whose set of n-cells is A^n/G. Show that the group G acts on the cellular chain complex $C_\bullet(X)$ and that $C_\bullet(X/G)$ is the invariant subcomplex.

   (b) Construct a CW structure on S^n with two cells in each dimension such that the cells are exchanged by the antipodal map. Write the associated chain complex and the action of $\mathbb{Z}/2\mathbb{Z}$ induced by the antipodal map. Compute the invariant subcomplex. You should find the chain complex for $\mathbb{RP}^n$, as indicated in part (a).

2. Recall that D^n is the unit closed ball centered at the origin in $\mathbb{A}^n$ and $\Delta^n \subset \mathbb{A}^{n+1}$ is the standard n-simplex defined by $t^1 + \cdots + t^{n+1} = 1$, $t^i \geq 0$. The boundary of D^n is S^{n-1}. In this exercise you will construct a standard orientation of the standard sphere.

   (a) Construct a homeomorphism $D^n \to \Delta^n$ as follows. First apply a translation and dilation to map D^n to the ball of radius $1/2$ centered at $(1/2, \ldots, 1/2)$. Compose with the map $(t^1, \ldots, t^n) \mapsto (t^1, \ldots, t^n, 1 - \Sigma t^i)$. Then apply a radial dilation centered at the point $(1/2, \ldots, 1/2, 1 - n/2)$.

   (b) We proved in class that the identity map $\Delta^n \to \Delta^n$ defines a generator of $H_n(\Delta^n, \partial\Delta^n)$. Compose with the inverse of the homeomorphism in (a) to produce a generator of $H_n(D^n, S^{n-1})$. Apply the boundary map to produce a generator of $\tilde{H}_{n-1}(S^{n-1})$. This is the standard generator.

3. Let R be a commutative ring and let M_1, M_2 be R-modules. Fix positive integers n, m.

   (a) Define $\mathrm{Hom}_R(M_1, M_2)$. It is an R-module. If $R = \mathbb{Z}$ then an R-module is an abelian group and we omit 'R' from the notation.

(b) Define the R-module $M_1 \otimes_R M_2$. If you have not seen this before you'll need to look it up. Describe it both by an explicit construction and characterize it by a universal property.

(c) Consider the short exact sequence

$$0 \longleftarrow \mathbb{Z}/n\mathbb{Z} \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z} \longleftarrow 0$$

where the map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by n. Apply $- \otimes \mathbb{Z}/m\mathbb{Z}$. What sequence do you obtain? Does it remain exact? If not, where does exactness fail and can you measure the failure of exactness?

(d) Now apply $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, -)$ and answer the same questions.

(e) Now apply $\mathrm{Hom}(-, \mathbb{Z}/m\mathbb{Z})$ and answer the same questions. In this case the direction of the arrows reverses.

4. A connected simply connected smooth compact 4-manifold X has a CW structure with a single 0-cell, a finite collection of 2-cells, and a single 4-cell. The homotopy type of X then depends on a single datum. What is it?