Problem Set # 8

M382E: Algebraic Topology

Due: October 28, 2008

These homeworks are an integral part of the course ...

Problems in Hatcher

Section 3.1 (page 204): 2, 3, 8, 10, 11

Other Problems

1. Let

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

be a short exact sequence of abelian groups. Let X be a space.

(a) Construct a long exact sequence of homology groups

$$\cdots \longrightarrow H_q(X; A') \longrightarrow H_q(X; A) \longrightarrow H_q(X; A'') \longrightarrow H_{q-1}(X; A') \longrightarrow \cdots$$

(b) Construct a long exact sequence of cohomology groups

$$\cdots \longrightarrow H^q(X; A') \longrightarrow H^q(X; A) \longrightarrow H^q(X; A'') \longrightarrow H^{q+1}(X; A') \cdots$$

(c) Write out the sequences for $X = \mathbb{RP}^2$, $X = \mathbb{RP}^3$, and the coefficient sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Do you understand all of the maps and how the sequence is exact?

2. Prove directly using a cochain complex (not the universal coefficient theorem) that for any space X (i) the cohomology group $H^1(X)$ is free and (ii) $H^1(X; A) \cong \mathrm{Hom}(H_1 X, A)$ for all abelian groups A.

3. (a) Construct a space X whose nonzero reduced integral cohomology groups are

$$\tilde{H}^1(X) \cong \mathbb{Z}, \qquad \tilde{H}^2(X) \cong \mathbb{Z}/2\mathbb{Z}, \qquad \tilde{H}^4(X) \cong \mathbb{Z}/3\mathbb{Z}, \qquad \tilde{H}^5(X) \cong \mathbb{Z}/2\mathbb{Z}$$

or prove that no such space exists.

(b) Same for

$$\tilde{H}^1(X;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \qquad \tilde{H}^2(X;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \qquad \tilde{H}^4(X;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}, \qquad \tilde{H}^5(X;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

4. (a) Construct a CW structure on $\mathbb{RP}^2 \times \mathbb{RP}^2$.

(b) Write the cellular chain complex for the CW structure. How is it related to the chain complex for $\mathbb{RP}^2$?

(c) Compute $H_\bullet(\mathbb{RP}^2 \times \mathbb{RP}^2)$.

(d) Compute $H_\bullet(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$.

(e) Compute $H^\bullet(\mathbb{RP}^2 \times \mathbb{RP}^2)$.

(f) Compute $H^\bullet(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$.