TECHNICAL REMARKS ON LECTURE #3 IN M382E

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Here are some technical remarks about bordism and the proof of Mayer-Vietoris for bordism. They involve ideas and techniques from differential topology. As you don't have a written account I am providing these notes.

Let X be a topological space. Recall that a bordism class in degree n is represented by a pair (M,ϕ), where M is a closed (=compact without boundary) n-manifold and $\phi\colon M\to X$ a continuous map. In lecture I defined an equivalence relation on such pairs. The definition I gave does not lend itself easily to gluing, so let's give a slightly more elaborate definition which is technically easier to work with. Once and for all fix $0<\epsilon<1/4$.

Definition 1. *A* bordism (W,τ,Φ) *from* (M,ϕ) *to* (M',ϕ') *consists of:* (i) *an* $(n+1)$*-manifold* W*,* (ii) *a smooth proper map* $\tau\colon W\to(-\epsilon,1+\epsilon)$*,* (iii) *a continuous map* $\Phi\colon W\to X$*,* (iv) *a diffeomorphism* $\tau^{-1}(-\epsilon,\epsilon)\approx(-\epsilon,\epsilon)\times M$ *with respect to which* $\Phi=\phi\circ\pi_2$*, and* (v) *a diffeomorphism* $\tau^{-1}(1-\epsilon,1+\epsilon)\approx(1-\epsilon,1+\epsilon)\times M'$ *with respect to which* $\Phi=\phi'\circ\pi_2$*.*

Here π_2 is projection onto the second factor of the product. I would draw a picture, but it is easier for me and more instructive for you to do so. You should draw W as a manifold with "cylinders" at each horizontal end and underneath draw the interval $(1-\epsilon,1+\epsilon)$ with the "time" map τ mapping down. What does your picture of a bordism look like in case M, M', or both are the empty n-manifold? By the way, recall that a proper map is one for which inverse images of compact sets are compact. I didn't give the diffeomorphisms in (iv) and (v) names—I won't refer to them explicitly in what follows—but you should keep them in mind when working through the details. A bordism according to Definition 1 is a bordism according to the cruder notion of bordism used in lecture: consider $\tau^{-1}[0,1]$.

As in lecture I will not mention orientations explicitly here. What I write applies to MSO as well as MO: I leave it to you to put in the orientations everywhere.

Now you can prove that the equivalence relation defined by bordism is transitive, which is one of the homework problems.

Suppose $U,V\subset X$ are open subsets. In lecture we defined the boundary map

$$\partial\colon MO_n(U\cup V)\longrightarrow MO_{n-1}(U\cap V) \tag{2}$$

as follows. Let (M,ϕ) represent a class in $MO_n(U\cup V)$. Then $\{\phi^{-1}(U),\phi^{-1}(V)\}$ is an open cover of M. Let $\{\rho_U,\rho_V\}$ be a subordinate partition of unity: ρ_U is supported on $\phi^{-1}(U)$, ρ_V is supported on $\phi^{-1}(V)$, $0\le\rho_U,\rho_V\le 1$, and $\rho_U+\rho_V=1$. By Sard we can find a regular value c of ρ_U with $\epsilon<c<1-\epsilon$. Set $N=\rho_U^{-1}(c)$. Then the pair $(N,\phi\big|_N)$ represents $\partial[M,\phi]$.

Claim. The map ∂ is independent of the choice of representative (M,ϕ) and of ρ_U, ρ_V, and c.

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Proof. Make another choice notated with primes. Then there is a bordism (W,τ,Φ) from (M,ϕ) to (M',ϕ') as in Definition 1. Choose a partition of unity $\{\sigma_U,\sigma_V\}$ subordinate to the open cover $\{\Phi^{-1}(U),\Phi^{-1}(V)\}$ of W. Let $\chi,\chi'\colon(-\epsilon,1+\epsilon)\to\mathbb{R}$ be smooth (cutoff) functions with $0\le\chi,\chi'\le1$; $\chi(\tau)=0$ for $\tau\le0$; $\chi(\tau)=1$ for $\tau\ge\epsilon$; $\chi'(\tau)=1$ for $\tau\le1-\epsilon$; $\chi'(\tau)=0$ for $\tau\ge1$. (Draw graphs!) Then define a new partition of unity on W by the formulas

$$\begin{aligned}\tilde\sigma_U&=(1-\chi\circ\tau)\rho_U+(\chi'\circ\tau)(\chi\circ\tau)\sigma_U+(1-\chi'\circ\tau)\rho'_U\\ \tilde\sigma_V&=(1-\chi\circ\tau)\rho_V+(\chi'\circ\tau)(\chi\circ\tau)\sigma_V+(1-\chi'\circ\tau)\rho'_V\end{aligned}\tag{3}$$

Consider the function $\tilde\sigma_U\times f\colon W\to\mathbb{R}\times\mathbb{R}$, where $f\colon W\to\mathbb{R}$ is a smooth function with $\epsilon\le f\le1-\epsilon$; $f=c$ on $\tau^{-1}(-\epsilon,\epsilon)$; and $f=c'$ on $\tau^{-1}(1-\epsilon,1+\epsilon)$. Let $\Delta\subset\mathbb{R}\times\mathbb{R}$ denote the diagonal and $C\subset W$ the closed subset $C=\tau^{-1}[-\epsilon,\epsilon]\ \amalg\ \tau^{-1}[1-\epsilon,1+\epsilon]$. Then $\tilde\sigma_U\times f$ is transverse to Δ on C. Apply the transversality extension theorem on page 72 of Guillemin-Pollack's *Differential Topology* to perturb $\tilde\sigma_U\times f$ to a map F which is transverse to Δ on all of W and agrees with $\tilde\sigma_U\times f$ on C. Let $Z=F^{-1}(\Delta)$. Then the triple $(Z,\tau\big|_Z,\Phi\big|_Z)$ is a bordism between the pairs $(N,\phi\big|_N)$ and $(N',\phi'\big|_{N'})$ computed as representatives of $\partial[M,\phi]$.

In the proof of Mayer-Vietoris I gave arguments for exactness in three different places. Those arguments left manifolds with corners, so either one needs to smooth the corners or use Definition 1. Fill in the details of the following sketches of rigorous arguments based on Definition 1.

(1) For exactness of

$$MO_n(U\cap V)\xrightarrow{i}MO_n(U)\oplus MO_n(V)\xrightarrow{j}MO_n(U\cup V)\tag{4}$$

suppose $(M,\phi)\ \amalg\ (M',\phi')$ maps to zero under j. Let (W,τ,Φ) be a null bordism with the image of Φ in $U\cap V$. Turn around the bordism by keeping W and Φ but constructing a new map $\tilde\tau$ which agrees with τ in the ϵ-neighborhood of M (measured by τ), agrees with $1-\tau$ in the ϵ-neighborhood of M', and interpolates otherwise. (Use a partition of unity.) Now take a partition of unity on W relative to the cover $\{\Phi^{-1}(U),\Phi^{-1}(V)\}$ as above; let c be a regular value away from 0,1; and then $(N,\Phi\big|_N)$ represents an element of $MO_n(U\cap V)$. You must now prove carefully that its image under i is bordant to $(M,\phi)\ \amalg\ (M',\phi')$. For this you use the bordisms $\{\rho_U\ge c\}\ \amalg\ \{\rho_U\le c\}$. But to construct the time function you will need to use the tubular neighborhood theorem and a partition of unity.

(2) For exactness of

$$MO_n(U)\oplus MO_n(V)\xrightarrow{j}MO_n(U\cup V)\xrightarrow{\partial}MO_{n-1}(U\cap V)\tag{5}$$

begin with (M,ϕ) in $U\cup V$ and do the boundary construction above to produce $(N,\phi\big|_N)$. If it represents the zero class choose a null bordism (W,τ,ψ) of (N,ϕ). In that construction let

$$\begin{aligned}M'&=\rho_U^{-1}[c,1]\\ M''&=\rho_U^{-1}[0,c]\end{aligned}\tag{6}$$

Note that $\dim W = n$. By gluing construct

$$\begin{aligned} P' &= M' \cup W \\ P'' &= M'' \cup W \end{aligned} \tag{7}$$

To do this gluing carefully you will need to take tubular neighborhoods in (6). Do so! Show that P' and P'' can be used to construct a class whose image under j is the original class represented by (M, ϕ).

(3) For exactness of

$$MO_{n+1}(U \cup V) \xrightarrow{\partial} MO_n(U \cap V) \xrightarrow{i} MO_n(U) \oplus MO_n(V) \tag{8}$$

begin with (M, ϕ) in $U \cap V$. This one is easier than the previous ones. You'll have two null bordisms which you need to glue, but as they come with open cylinders, there is no further work necessary to carry out the gluing.