Problem Set # 3

M392C: Topics in Geometry and Physics

1. Suppose M is a smooth manifold and $\Delta \subset TM$ a distribution. Define

$$\mathcal{I}(\Delta) = \{\omega \in \Omega^\bullet(M) : \omega\big|_\Delta = 0\}.$$

   (a) Prove that $\mathcal{I}(\Delta) \subset \Omega^\bullet(M)$ is an ideal.

   (b) Prove that if Δ has corank r—that is, if $\dim \Delta_m + r = \dim_m M$ for all $m \in M$—then Δ is locally generated by r independent 1-forms.

   (c) Prove that $\mathcal{I}(\Delta)$ is closed under d if and only if Δ is integrable.

   (d) Consider the distribution Δ on $\mathbb{A}^3$ spanned by the vector fields $\partial/\partial x$ and $x\,\partial/\partial y + \partial/\partial z$. Show that Δ is not integrable. Show that any point $(x, y, z) \in \mathbb{A}^3$ may be joined to $(0, 0, 0)$ by a piecewise smooth curve whose tangent line belongs to Δ.

2. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra of left invariant vector fields. For $g \in G$ define $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ as the action of $L_g R_{g^{-1}}$ on left invariant vector fields, where L_g (R_g) is left (right) translation by g. Check this is well-defined and determines a homomorphism $G \to \mathrm{End}(\mathfrak{g})$. Now for $\xi \in \mathfrak{g}$ define $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ as $d(\mathrm{Ad}_{g(t)})/dt$ at $t = 0$ for any curve with $g(0) = e$ and $\dot{g}(0) = \xi(e)$. Prove that $\xi \mapsto \mathrm{ad}_\xi$ is a homomorphism $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ of Lie algebras and that $\mathrm{ad}_\xi(\eta) = [\xi, \eta]$ for all $\xi, \eta \in \mathfrak{g}$.

3. (a) Let V be an n-dimensional real vector space and $\mathcal{B}(V)$ the right $GL_n(\mathbb{R})$-torsor of bases. Let Θ^i_j be the Maurer-Cartan forms in the standard basis of the Lie algebra of $GL_n(\mathbb{R})$. Suppose $b(t)$ is a smooth curve in $\mathcal{B}(V)$. Write the basis $b(t)$ as $\{e_1(t), \dots, e_n(t)\}$ and the dual basis as $\{e^1(t), \dots, e^n(t)\}$. Prove that

$$\Theta^i_j(\dot{b}) = \langle e^i(0), \dot{e}_j(0)\rangle.$$

   (b) Let A be an n-dimensional real affine space and $\mathcal{B}(A)$ the right $\mathrm{Aff}_n(\mathbb{R})$-torsor of bases of the underlying vector space at all points of A. Let θ^i, Θ^i_j be the Maurer-Cartan forms in the standard basis of the Lie algebra of $\mathrm{Aff}_n(\mathbb{R})$. (Define this!) Suppose $b(t)$ is a smooth curve in $\mathcal{B}(A)$ which projects to the curve $x(t)$ in A, and write the underlying basis of V as in (a). Prove that

$$\theta^i(\dot{b}) = \langle e^i(0), \dot{x}(0)\rangle.$$

4. Let G be a Lie group with Maurer-Cartan form θ. Compute $R_g^*\theta$ for $g \in G$. Do this first for a matrix group, where you can write $\theta = A^{-1}dA$ for $A\colon G \to M_N(\mathbb{R})$ the natural matrix-valued function on a matrix group. ($M_N(\mathbb{R})$ is a vector space, so the differential of the function A is defined as a $M_N(\mathbb{R})$-valued 1-form.)

5. Let G be a Lie group with Maurer-Cartan form θ.

   (a) Suppose $\xi\colon [0,L] \to \mathfrak{g}$ is a time-varying left invariant vector field on G. Let $\tilde{\theta} = \xi(t)\,dt$ be a $\mathfrak{g}$-valued 1-form on $[0,L]$. Show that a solution $g\colon [0,L] \to G$ to the equation $g^*\theta = \tilde{\theta}$ is an integral curve for ξ.

   (b) Replace $[0,L]$ by $\mathbb{R}$ and let ξ be constant. Prove that the unique solution $g\colon \mathbb{R} \to G$ with $g(0) = e$ is a 1-parameter group: $g(t_1 + t_2) = g(t_1)g(t_2)$.

   (c) Suppose $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra. Construct a connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$. Is $H \subset G$ necessarily closed?

6. Let $\Sigma \subset \mathbb{E}^3$ be a surface in Euclidean space. Replace Σ by a sufficiently small open subset so that there is a global framing of $\mathbb{E}^3$ along Σ. Choose a framing $\{e_1, e_2, e_3\}\colon \Sigma \to \mathcal{B}_O(\mathbb{R}^3)$ and combine with the embedding $\Sigma \subset \mathbb{E}^3$ to obtain an orthonormal framing $\Sigma \to \mathcal{B}_O(\mathbb{E}^3)$. What can you say about the pullbacks of the Maurer-Cartan forms $\theta^1, \theta^2, \theta^3, \Theta^1_2, \Theta^1_3, \Theta^2_3$. (See problem 3.) Can you recognize them in terms of classical differential geometry? What do the Maurer-Cartan equations tell?

7. Let V be a 4-dimensional complex vector space and fix a nonzero volume form $\mu \in \bigwedge^4 V^*$. Define a $\mathbb{C}$-valued bilinear form b on $\bigwedge^2 V$ by $b(\alpha, \beta) = \langle \mu, \alpha \wedge \beta \rangle$ for $\alpha, \beta \in \bigwedge^2 V$.

   (a) Choose a basis $\{e_1, e_2, e_3, e_4\}$ of V and the induced basis $\{e_i \wedge e_j : i < j\}$ of $\bigwedge^2 V$. Then choose $\mu = e^1 \wedge e^2 \wedge e^3 \wedge e^4$, where $\{e^1, e^2, e^3, e^4\}$ is the dual basis. Show that b is nondegenerate.

   (b) Define the homomorphism

   $$\pi\colon\ \mathrm{Aut}(V, \mu) \longrightarrow \mathrm{Aut}(\textstyle\bigwedge^2 V, b)$$

   which maps a volume-preserving automorphism of V to a bilinear form-preserving automorphism of $\bigwedge^2 V$. Write the corresponding map of Lie algebras. Prove that the latter is an isomorphism.

   (c) Deduce that the image of π is open and a subgroup, whence π maps onto the identity component of $\mathrm{Aut}(\bigwedge^2 V, b)$. (Show that the latter group has two components.)

   (d) Prove that the kernel of π is $\{\pm\,\mathrm{id}\}$ so that π is a 2:1 covering. Deduce that $SL_4(\mathbb{C})$ is a double covering of $SO_6(\mathbb{C})$.

(e) Now choose $\omega \in \bigwedge^2 V^*$ with $(\omega \wedge \omega)/2 = \mu$. For example, take $\omega = e^1 \wedge e^2 + e^3 \wedge e^4$. Let $W \subset \bigwedge^2 V$ be the annihilator of ω. Then use the restriction of the map π above to define

$$\pi\colon \operatorname{Aut}(V, \omega) \longrightarrow \operatorname{Aut}(W, b).$$

As before, prove that π is a 2:1 covering map. Deduce that $Sp_4(\mathbb{C})$ is a double covering of $SO_5(\mathbb{C})$.

(f) Write $V = U_1 \oplus U_2$ as the direct sum of 2-dimensional subspaces. Choose nonzero $\omega_1 \in \bigwedge^2 U_1^*$ and $\omega_2 \in \bigwedge^2 U_2^*$. Construct a decomposition $\bigwedge^2 V^* \cong \bigwedge^2 U_1^* \;\oplus\; \bigwedge^2 U_2^* \;\oplus\; U_1^* \otimes U_2^*$. Then, following the ideas in previous parts, construct a 2:1 covering

$$\pi\colon \operatorname{Aut}(U_1, \omega_1) \times \operatorname{Aut}(U_2, \omega_2) \longrightarrow \operatorname{Aut}(U_1^* \otimes U_2^*, b).$$

(In the definition of b take $\mu = \omega_1 \wedge \omega_2$.) Deduce that $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ is a double covering of $SO_4(\mathbb{C})$.

(g) Finally, construct a double covering $\pi\colon SL_2(\mathbb{C}) \to SO_3(\mathbb{C})$.