Problem Set # 4

M392C: Topics in Geometry and Physics

1. Let $\pi\colon E \to M$ be a proper submersion of smooth manifolds with M connected. Prove that π is a fiber bundle.

2. Let G be a Lie group.

   (a) Show that the automorphism group $\mathrm{Aut}(T)$ of a G-torsor is the Lie group associated to T by the conjugation action of G on itself. Why is that associated space a Lie group?

   (b) Show that the Lie algebra of the group in (a) is the Lie algebra associated to T by the adjoint action $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$. Why is that associated space a Lie algebra?

   (c) Suppose $\pi\colon P \to M$ is a principal G-bundle. Construct associated bundles $G_P \to M$, $\mathfrak{g}_P \to M$ of Lie groups and Lie algebras. The latter is the *adjoint bundle*. Show that the group $\mathrm{Aut}_0 P$ of gauge transformations is the group of sections of $G_P \to M$ and its Lie algebra is $\Omega^0_M(\mathfrak{g}_P)$, the group of sections of the adjoint bundle.

3. Let $\pi\colon P \to M$ be a principal G-bundle and $\rho\colon G' \to G$ a homomorphism of Lie groups. A *reduction of P to G'* is a pair consisting of a principal G'-bundle $Q \to M$ and a map $\varphi\colon Q \to P$ which covers id_M and intertwines ρ: $\varphi(qg') = \varphi(q)\rho(g')$ for all $q \in Q$, $g' \in G'$.

   (a) Prove that if ρ is injective, then a reduction is equivalent to a section of the associated bundle with fiber G/G'. (First define this associated bundle.) What does this say if $G' = \{e\}$?

   (b) Use covering space theory to analyze the reduction problem when G', G are discrete groups. When is a lift defined? Can you determine an obstruction which measures whether a lift exist? What are the equivalence classes of lifts? You may want to try the case when ρ is injective with abelian cokernel, or the case when ρ is surjective with abelian kernel.

4. Let $E \to M$ be a vector bundle of rank r. Define a principal $GL_r(\mathbb{R})$-bundle $\pi\colon \mathcal{B}(E) \to M$ which deserves to be called the *bundle of frames*. Show that $E \to M$ is associated to it via the standard representation. Show that a metric on E corresponds to a reduction of structure group to $O_r \subset GL_r(\mathbb{R})$. Similarly, define an orientation, complex structure, and symplectic structure in terms of reduction of structure groups.

5. (a) If G is a Lie group and $H \subset G$ a closed subgroup, verify that $\pi\colon G \to G/H$ is a principal H-bundle. (You can consult Warner's book for some details.) Prove that the tangent bundle of G/H is the associated bundle via the adjoint representation $\mathrm{Ad}\colon H \to \mathrm{Aut}(\mathfrak{g}/\mathfrak{h})$ on the quotient of Lie algebras.

(b) Show that there is a left G-action by automorphisms on the principal bundle in (a). This is called a *homogeneous* bundle.

(c) Write the Hopf bundle, which is a principal $\mathbb{T}$-bundle over S^2, as a homogeneous bundle. What is the total space?

(d) Construct a homogeneous SU_2-bundle over S^4. Construct a homogeneous SO_4-bundle over S^4. What is the total space of each? Can you find a 3-sphere bundle over S^4 whose total space is the 7-sphere?

6. (a) Suppose P is a Riemannian manifold and G a Lie group acting freely on the right by isometries such that the quotient $\pi\colon P \to M$ is a principal G-bundle. Show that the orthogonal subspaces to the fibers form a connection.

(b) As a warmup to complex notation, consider the complex line $\mathbb{C}$ with coordinate z as a real manifold $\mathbb{R}^2$. Then $\partial/\partial z + \partial/\partial \bar{z}$ is a vector field on $\mathbb{C}$. Which? What about $i[\partial/\partial z - \partial/\partial \bar{z}]$?

(c) Now consider the unit sphere in $\mathbb{C}^2$ with its round metric inherited from the standard metric on $\mathbb{C}^2$ and the action of the unit norm complex numbers $\mathbb{T}$ by isometries: $(z^1, z^2)\cdot\lambda = (z^1\lambda, z^2\lambda)$. (The standard real inner product may be written as

$$\langle (z^1, z^2), (w^1, w^2)\rangle = \mathrm{Re}[\overline{z^1}w^1 + \overline{z^2}w^2].$$

What is the quotient space? At each point of the sphere write a tangent to the orbit, so a basis for the vertical tangent space. Better, construct the isomorphism of that vertical tangent space with the Lie algebra of $\mathbb{T}$. Find a basis of the horizontal tangent space. Show that the connection is not integrable. Can you write the connection 1-form and compute its curvature?

7. Let V be a real or complex finite dimensional vector space and $k \le \dim V$ a positive integer. Recall the Stiefel manifold $St_k(V)$ of injections $b\colon \mathbb{R}^k \to V$ and its free right $GL_k(\mathbb{R})$-action which defines a principal bundle $\pi\colon St_k(V) \to Gr_k(V)$ over the Grassmannian. Write the tangent bundle $T\big(Gr_1(V)\big)$ as associated to this principal bundle. Is the tangent bundle to $Gr_k(V)$ an associated bundle for $k > 1$?