Problem Set # 5

M392C: Topics in Geometry and Physics

Throughout G is a Lie group with Lie algebra $\mathfrak{g}$.

1. Let $\pi\colon P \to M$ be a principal G-bundle.
   (a) Suppose $\alpha \in \Omega^1_M$. Show that $\beta = \pi^*\alpha$ satisfies $\iota_\eta \beta = 0$ for all vertical vectors β and $R_g^*\beta = \beta$ for all $g \in G$. Conversely, show that if a 1-form $\beta \in \Omega^1_P$ satisfies these two properties, then $\beta = \pi^*\alpha$ for a unique $\alpha \in \Omega^1_M$.
   (b) Repeat the exercise for a q-form for arbitrary $q \geq 0$. Is the same assertion true?

2. Let $\pi\colon P \to M$ be a G-bundle with connection Θ. Suppose M is connected and we fix a basepoint $p \in P$. Set $m = \pi(p)$. Then to each loop based at m the holonomy assigns an element of g. Prove that the group $\mathrm{Aut}(\Theta)$ of gauge transformations φ which fix Θ is isomorphic to the largest closed subgroup of G consisting of elements which commute with all holonomies.

3. Let $E \to M$ be a vector bundle with a *flat* connection, i.e., with vanishing curvature. Then the extended de Rham complex
$$0 \longrightarrow \Omega^0_M(E) \xrightarrow{d_\nabla} \Omega^1_M(E) \xrightarrow{d_\nabla} \Omega^2_M(E) \longrightarrow \cdots$$
is a complex: $d_\nabla^2 = 0$. The cohomology of this complex is denoted $H^\bullet(M;E)$ and is a *twisted* de Rham cohomology of M. Construct a nontrivial flat covariant derivative on a real line bundle over S^1 whose total space is a Möbius band. (Hint: it is associated to a principal $\mathbb{Z}/2\mathbb{Z}$-bundle, i.e., to a double cover.) Compute the twisted de Rham cohomology for this bundle.

4. (a) Let ∇ be a covariant derivative on the tangent bundle $TM \to M$ of a manifold. The *torsion* T_∇ is a tensor $T_\nabla\colon \bigwedge^2 TM \to TM$ defined on vector fields ξ_1, ξ_2 by the formula
$$T_\nabla(\xi_1, \xi_2) = \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1, \xi_2].$$
   Check that this is tensorial, i.e., is linear over functions.
   (b) Now if $E \to M$ is a real vector bundle with inner product $\langle -, - \rangle$, then the covariant derivative is *orthogonal* if for any vector fields ξ_1, ξ_2 we have
$$d\langle \xi_1, \xi_2 \rangle = \langle \nabla\xi_1, \xi_2 \rangle + \langle \xi_1, \nabla\xi_2 \rangle.$$
   Prove that this is the case if and only if the associated parallel transport preserves the metric.
   (c) Let M be a Riemannian manifold. Prove that there is a unique covariant derivative on the tangent bundle which is orthogonal and torsionfree ($T_\nabla = 0$). This is the *Levi-Civita covariant derivative*. Hint: Derive a formula for $\langle \nabla_{\xi_1}\xi_2, \xi_3 \rangle$ by using the two equations above.

5. Let M be a smooth n-manifold and $\mathcal{B}(M) \to M$ its principal $GL_n(\mathbb{R})$-bundle of frames. Fix a connection on this bundle.

   (a) Construct a global framing of the manifold $\mathcal{B}(M)$. More precisely, find an isomorphism of each tangent space with $\mathbb{R}^n \oplus \mathfrak{g}$.

   (b) Dually, find global 1-forms θ^i, Θ^i_j on $\mathcal{B}(M)$ which at each point form a basis for the cotangent space. Here $1 \le i, j \le n$. (Hint: the connection form is $\Theta = (\Theta^i_j)$.)

   (c) Derive the structure equations

$$\begin{aligned} d\theta^i + \Theta^i_j \wedge \theta^j &= \tau^i \\ d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j &= \Omega^i_j, \end{aligned}$$

   where $\tau = (\tau^i)$ is the torsion and $\Omega = (\Omega^i_j)$ is the curvature. Compare with the structure equations of the affine group (Problem Set #2).

6. (a) Let V be a vector space with inner product and $k \ge 1$ an integer. Recall the principal $GL_k(\mathbb{R})$-bundle $\pi\colon St_k(V) \to Gr_k(V)$ whose total space is the Stiefel manifold of injections $b\colon \mathbb{R}^k \to V$. (You may refer to #6 and #7 on the previous problem set.) Identify the tangent space $T_b St_k(V)$. (Hint: consider $b^*\dot{b}$, where $\dot{b}$ is the tangent to a curve b_t.) Use the metric on V to construct a natural Riemannian metric on $St_k(V)$. Show that $GL_k(\mathbb{R})$ acts by isometries. The orthogonal complements to the verticals then define a connection.

   (b) Show that the covariant derivative on the associated tautological rank k vector bundle is the one constructed in lecture, compressed from the trivial covariant derivative on $Gr_k(V) \times V \to Gr_k(V)$.

7. (a) Let $Q \to [0,1] \times M$ be a principal G-bundle. Choose a connection on Q. (Why does it exist?) Use parallel transport to construct an isomorphism $Q\big|_{\{0\}\times M} \to Q\big|_{\{1\}\times M}$.

   (b) Let $f_t\colon M \to N$ be a smooth homotopy and $P \to N$ a principal bundle. Prove that $f_0^*P \cong f_1^*P$.

   (c) Prove that a principal bundle over a contractible manifold is trivializable.

   (d) Classify up to isomorphism all principal $\mathbb{T}$-bundles on S^n for all n.