Problem Set # 6

M392C: Topics in Geometry and Physics

1. Let V be a complex vector space.

   (a) Prove that any linear map $T\colon V \to V$ has a fixed line, i.e., an eigenspace.

   (b) Let $B\colon V \times V \to V$ be a nondegenerate bilinear form. Suppose T is self-adjoint or skew-adjoint:

   $$B(Tv, w) = \pm B(v, Tw), \qquad v, w \in V.$$

   Prove that T is diagonalizable: there is a basis of eigenvectors $\{v_i\} \subset V$ with $Tv_i = \lambda v_i$. (Hint: Find one eigenvector and consider the orthogonal subspace with respect to B.)

   (c) Suppose $\{T_\alpha\}_{\alpha \in A}$ is a set (not necessarily finite or countable) of pairwise commuting endomorphisms of V, each of which is diagonalizable. Prove that they are simultaneously diagonalizable. In other words, there is a basis $\{v_i\} \subset V$ and functions $\lambda_i\colon A \to \mathbb{C}$ such that $T_\alpha(v_i) = \lambda_i(\alpha)v_i$.

2. Let $\mathbb{A}^2$ be the affine plane and M the space of configurations of n unordered points for some $n \in \mathbb{Z}^{>0}$. Thus

   $$M = \bigl\{\{p_1, \dots, p_n\} \subset \mathbb{A}^2 : p_i \neq p_j \text{ if } i \neq j\bigr\}.$$

   Let $P \subset (\mathbb{A}^2)^{\times n}$ be the space of *ordered* n-tuples of distinct points in the plane with the subspace topology. Identify P as the space of injective maps $p\colon \{1, \dots, n\} \to \mathbb{A}^2$. Define a free action of the symmetric group Sym_n with quotient $\pi\colon P \to M$ and topologize M with the subspace topology. What is the automorphism group of the principal Sym_n-bundle $\pi\colon P \to M$?

3. Let X be a closed oriented connected 2-manifold of genus g and $x \in X$ a basepoint. Let G be a finite group. Recall the Frobenius formula

   $$\frac{1}{\#G} \# \operatorname{Hom}\bigl(\pi_1(X, x), G\bigr) = \sum_{\chi} \left(\frac{\chi(e)}{\#G}\right)^{2-2g},$$

   where the sum is over the characters χ of irreducible representations of G and $e \in G$ is the identity element.

   (a) Check this formula directly for G a cyclic group. For G an abelian group.

   (b) Determine the irreducible representations and characters for the nonabelian group Sym_3 of order 6. (A representation of G on V is irreducible if every G-invariant subspace $W \subset V$ is either $W = 0$ or $W = V$.)

4. Let A, B be groupoids. Recall that a map $\alpha\colon A \to B$, which we'll term a *1-morphism*, is a pair of maps $\alpha_0\colon A_0 \to B_0$, $\alpha_1\colon A_1 \to B_1$ which commute with all of the structure maps of the groupoid. Let $\alpha, \beta\colon A \to B$ be 1-morphism. A *2-morphism* $\theta\colon \alpha \to \beta$ is a map $\theta\colon A_0 \to B_1$ where for $a \in A_0$ we have $\big(\alpha(a) \xrightarrow{\theta(a)} \beta(a)\big) \in B_1$ and such that for all $(a_0 \xrightarrow{f} a_1) \in A_1$ the diagram

$$\begin{array}{ccc} \alpha(a_0) & \xrightarrow{\alpha(f)} & \alpha(a_1) \\ {\scriptstyle \theta(a_0)}\downarrow & & \downarrow{\scriptstyle \theta(a_1)} \\ \beta(a_0) & \xrightarrow{\beta(f)} & \beta(a_1) \end{array}$$

commutes.

(a) Construct a composition law which composes two 2-morphisms and produces a 2-morphism. Construct a composition law which composes a 1-morphism and a 2-morphism (on either side) to produce a 2-morphism. When are such compositions defined? Investigate the notion of associativity.

(b) An equivalence of groupoids is a pair of maps $\alpha\colon A \to B$, $\beta\colon B \to A$ and *invertible* 2-morphisms $\mathrm{id}_A \to \beta\alpha$ and $\mathrm{id}_B \to \alpha\beta$. Construct an equivalence between $G/\!/G$ (G acts by conjugation) and $*/\!/G \cup */\!/(\mathbb{Z}/2\mathbb{Z}) \cup */\!/(\mathbb{Z}/3\mathbb{Z})$, where $G = \mathrm{Sym}_3$ is the symmetric group on 3 letters.

(c) For G a discrete group and X a connected topological space with basepoint $x \in X$, construct an equivalence between the groupoid $\mathrm{Bun}_G(X)$ of principal G-bundles and $\mathrm{Hom}\big(\pi_1(X, x), G\big)/\!/G$, where G acts by conjugation.

5. Let S be a finite set with a finite group G acting and let $\beta\colon S/\!/G \to *$ be the unique map to a point. Prove that the pushforward of a function φ on $S/\!/G$, which is a G-equivariant function on S, is the number

$$\beta_* \varphi = \frac{1}{\#G} \sum_{s \in S} \varphi(s).$$

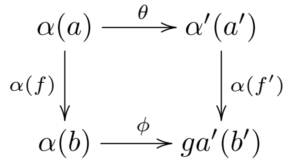
6. Let A, A', B be groupoids and

$$\begin{array}{ccc} & & A' \\ & & \downarrow{\scriptstyle \alpha'} \\ A & \xrightarrow{\alpha} & B \end{array}$$

a diagram of morphisms. Define the pullback

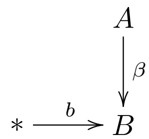
$$\begin{array}{ccc} C & \overset{\gamma'}{\dashrightarrow} & A' \\ {\scriptstyle \gamma}\,\vdots & & \downarrow{\scriptstyle \alpha'} \\ A & \xrightarrow{\alpha} & B \end{array}$$

as follows. Objects of C are triples $(a, a', \theta) \in A_0 \times A'_0 \times B_1$ where $\theta\colon \alpha(a) \to \alpha'(a')$. A morphism $(a, a', \theta) \to (b, b', \phi)$ in C is a pair $(f, f') \in A_1 \times A'_1$ such that $f\colon a \to B$, $f'\colon a' \to b'$, and the diagram



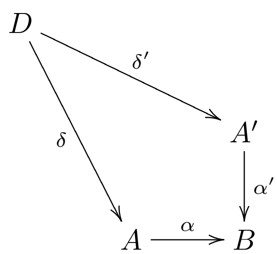
commutes.

(a) Suppose $\beta\colon A \to B$ is a map of finite groupoids and $b \in B_0$ an object. Show that the pullback of

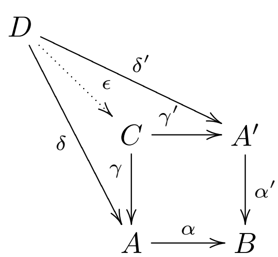


is the fiber of β over b defined in lecture.

(b) Suppose that



is a commutative diagram in the sense that we are given a natural *isomorphism* $\theta\colon \alpha\delta \to \alpha'\delta'$. Prove that it factors through the pullback:



What precisely do we mean by the commutativity of the diagram? What kind of uniqueness do we have?

(c) Formulate the universal property satisfied by the pullback in this context. What uniqueness is satisfied?