Problem Set # 8

M392C: Topics in Geometry and Physics

1. Imitate the proof of the Darboux theorem to prove the following. Let M be a smooth manifold and μ a volume form on M, i.e., a nonvanishing differential form of top degree. Then around every point of M there is a local coordinate system $x^1, \dots, x^n$ such that $\mu = dx^1 \wedge \cdots \wedge dx^n$.

2. Write the equation(s) for an integral curve of the symplectic gradient of a function H on $\mathbb{A}^{2m}$ with symplectic form $dp_i \wedge dq^i$.

3. (a) Let M^{2m} be a symplectic manifold with symplectic form ω. Prove that $\omega^m/m!$ is a volume form, i.e., is nonzero. (This is a linear algebra exercise: the nonvanishing of $\omega^m/m!$ is equivalent to the nondegeneracy of ω.)

   (b) Deduce that if M is compact, then $\int_M \omega^m/m!$ is nonzero. Conclude that the de Rham cohomology class of ω and all of its powers is nonzero.

   (c) Prove that $S^1 \times S^3$ does not admit a symplectic structure.

4. Let M be the two-dimensional torus, regarded as an abelian Lie group, with its invariant parallelism and an invariant symplectic structure. Show that a nonzero parallel vector field is not the symplectic gradient of a function on M.

5. Let M be a smooth manifold.

   (a) Let $\pi\colon T^*M \to M$ be the cotangent bundle of M. Define a 1-form θ on T^*M by $\theta_\alpha(\xi) = \alpha(\pi_*\xi)$, where $\alpha \in T^*M$ and $\xi \in T_\alpha(T^*M)$. Show that $\omega = d\theta$ is a symplectic form on T^*M.

   (b) Let $x^1, \dots, x^n$ be a local coordinate system on an open set $U \subset M$. Define a coordinate system $x^1, \dots, x^n, p_1, \dots, p_n$ on $\pi^{-1}(U)$ by specifying that a 1-form $\alpha = p_i(\alpha)dx^i$. Write θ and ω in this coordinate system.

   (c) Let $f\colon M \to \mathbb{R}$ be a smooth function. What is the symplectic gradient of π^*f?

   (d) Let ξ be a smooth vector field on M. Then ξ defines a smooth function on T^*M (which is linear on each fiber). What is its symplectic gradient? What is the flow generated?

6. Let M be a Riemannian manifold with metric $\langle -, - \rangle$. Use the metric to construct an isomorphism of vector bundles $TM \to T^*M$. Pullback the symplectic form on T^*M to TM. What is the symplectic gradient of the function $\xi \to \frac{1}{2}\langle \xi, \xi \rangle$ on TM? What is the flow generated?