Problem Set # 9

M392C: Topics in Geometry and Physics

1. Let M, X be smooth manifolds, ∇ the covariant derivative of a linear connection on X, and $\phi\colon M \to X$ a smooth map. Write T_∇ for the torsion. Prove that $d_\nabla d\phi = \phi^* T_\nabla$. The first point is to carefully interpret that equation. Namely, $d\phi \in \Omega^1_M(\phi^* TX)$ and then d_∇ acts as the extension of the de Rham complex to forms with values in ϕ^*TX using ∇.

2. Let $V \to M$ be a real vector bundle of rank r with associated $GL_r(\mathbb{R})$-bundle of frames $\mathcal{B}(V) \to M$. Define the *orientation bundle* $\mathfrak{o}_V \to M$ to be the real line bundle associated to the representation $g \mapsto \operatorname{sign}\det(g)$ of $GL_r\mathbb{R}$ on $\mathbb{R}$.

   (a) What is the relation to the double cover $\mathcal{B}(V)/GL_r^+(\mathbb{R}) \to M$, where $GL_r^+(\mathbb{R}) \subset GL_r(\mathbb{R})$ is the index two subgroup of linear maps with positive determinant?

   (b) Show that an orientation of V determines a trivialization of $\mathfrak{o}_V \to M$.

   (c) Given a short exact sequence of vector bundles

   $$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

   over M, construct a canonical isomorphism $\mathfrak{o}_V \cong \mathfrak{o}_{V'} \otimes \mathfrak{o}_{V''}$. (Hint: Consider the subbundle of frames of V such that the first rank V' vectors form a basis of V'.)

   (d) Define $\mathfrak{o}_M = \mathfrak{o}_{TM}$. Recall the definition $\Omega_M^{|-q|} := \Omega_M^{n-q}(\mathfrak{o}_M)$ if $\dim M = n$. Prove that elements of $\Omega_M^{|-q|}$ are sections of $\bigwedge^q TM \otimes \operatorname{Dens}(M)$, where $\operatorname{Dens}(M) \to M$ is the line bundle associated to the representation $g \mapsto |\det(g)|$ of $GL_n(\mathbb{R})$.

   (e) Suppose $i\colon N \hookrightarrow M$ is a codimension q submanifold with oriented normal bundle. Use (c) to construct an isomorphism $i^*\mathfrak{o}_M \cong \mathfrak{o}_N$. Show, then, that elements of $\Omega_M^{|-q|}$ restrict to densities on N.

3. Repeat the example of a free nonrelativistic particle on a Riemannian manifold X where you add a potential function $U\colon X \to \mathbb{R}$. Recall that we work on $\operatorname{Map}(\mathbb{R}, X) \times \mathbb{R}$ with differential $D = \delta + d$, and we orient $\mathbb{R}$ to identify densities with 1-forms and $|-1|$-forms with 0-forms. Then define

   $$L = \{\frac{1}{2}\langle \dot{x}, \dot{x}\rangle - U \circ x\}\, dt.$$

   What is the corresponding variational 1-form γ? Compute $\delta L + d\gamma$ and so the equations of motion (Newton's law). Let $\mathcal{M}$ be the space of solutions and compute the symplectic form $\delta\gamma$ on $\mathcal{M}$.

4. Let V be a finite-dimensional real vector space endowed with a nondegenerate symmetric real-valued bilinear form. We denote the pairing of vectors v, w as $\langle v, w \rangle$. Let $n = \dim V$.

   (a) Define an induced nondegenerate symmetric form on the dual V^* and on all exterior powers of both V and V^*.

   (b) The highest exterior power of a vector space is one-dimensional, and is called the *determinant line* of the vector space. An element in the determinant line $\operatorname{Det} V^*$ is a *volume form* on V. There are precisely two such forms ω such that $\langle \omega, \omega \rangle = \pm 1$; they are opposite. Choose one. This amounts to fixing an *orientation* of V, which is a choice of component of $\operatorname{Det} V \setminus \{0\}$. Then the *Hodge $*$ operator*, which is a map

   $$*\colon \bigwedge^q V^* \longrightarrow \bigwedge^{n-q} V^*$$

   is defined implicitly by the equation

   $$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, \omega, \qquad \alpha, \beta \in \bigwedge^q V^*.$$

   Verify that the $*$ operator is well-defined.

   (c) Show that without choosing an orientation of V there is a Hodge $*$ operator

   $$*\colon \bigwedge^q V^* \longrightarrow \bigwedge^{|-q|} V^* := \bigwedge^{n-q} V^* \otimes \mathfrak{o}_V.$$

   (d) Compute $**$. The answer depends on the signature of the quadratic form and the degree q.

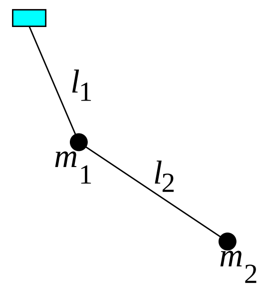
   (e) Compute $*$ on a 3 dimensional vector space with a positive definite inner product (choose a basis!) and on a 4 dimensional vector space with a Lorentz inner product.

5. Write the lagrangians for the following mechanical systems with one or more particles; you may also want to work out the equations of motion. These systems are placed in a uniform gravitational field. The potential energy (determined only up to a constant) for a particle of mass m at height h in the gravitational field is mgh for some universal constant g. These problems are taken from *Mechanics*, by Landau and Lifshitz, a highly recommended text.

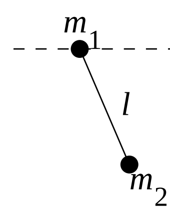
   (a) A simple pendulum of mass m and length ℓ.

   (b) A coplanar double pendulum.

   (c) A simple pendulum of mass m_2, with a mass m_1 at the point of support which can move on a horizontal line lying in the plain in which m_2 moves.



l_1
m_1
l_2
m_2



m_1
l
m_2