Cusps of Hilbert modular varieties

D. B. McReynolds*

March 15, 2006

Abstract

Let M denote a virtual n–torus bundle over an m–torus. In this article we give a necessary and sufficient condition for M to be diffeomorphic to a cusp cross-section of a Hilbert modular variety. One application of this classification theorem answers an implicit question of Hirzebruch on the possible isomorphism types of the fundamental group of a cusp cross-section of a Hilbert modular variety. Specialized to Hilbert modular surfaces, this proves that every Sol 3–manifold is diffeomorphic to a cusp cross-section of a (generalized) Hilbert modular surface. We conclude this article by proving that certain Sol 3–manifolds cannot arise as a cusp cross-section of a 1–cusped nonsingular Hilbert modular surface.

1 Introduction

Main results

Let k be a totally real number field with $[k:\mathbf{Q}] = n$, $\mathscr{O}_k$ the ring of integers of k, and $\sigma_1, \dots, \sigma_n$ denote the n real embeddings of k. The group $\mathrm{PSL}(2;\mathscr{O}_k)$ is an arithmetic subgroup of the n–fold product $(\mathrm{PSL}(2;\mathbf{R}))^n$ (see [5]) via the embedding $\xi \longmapsto (\sigma_1(\xi), \dots, \sigma_n(\xi))$ for $\xi \in \mathrm{PSL}(2;\mathscr{O}_k)$. Through this embedding, $\mathrm{PSL}(2;\mathscr{O}_k)$ acts with finite volume on the n–fold product of real hyperbolic planes $(\mathbf{H}^2_{\mathbf{R}})^n$. The group $\mathrm{PSL}(2;\mathscr{O}_k)$ is called *the Hilbert modular group*. More generally, we call any subgroup Λ of $\mathrm{PSL}(2;k)$ which is commensurable with $\mathrm{PSL}(2;\mathscr{O}_k)$ a *Hilbert modular group* and the quotients $(\mathbf{H}^2_{\mathbf{R}})^n/\Lambda$, *Hilbert modular varieties*. In the case that k is a real quadratic number field, these quotients are called *Hilbert modular surfaces*. For more on Hilbert modular surfaces, see [10] or [26].

The primary focus of this article is cusp cross-sections of Hilbert modular varieties. These manifolds are virtual n–torus bundles over $(n-1)$–tori where $[k:\mathbf{Q}] = n$ and

*Supported by a V.I.G.R.E. graduate fellowship and Continuing Education fellowship.

rank $\mathcal{O}_k^\times = n-1$. For brevity, we simply call these *virtual* $(n,n-1)$*–torus bundles*. Recall that an n–torus bundle over an m–torus is the total space of a fiber bundle with base manifold T^m and fiber T^n (see [24]). We call such manifolds simply (n,m)*–torus bundles*. We say that N is a *virtual* (n,m)*–torus bundle* if N is finitely covered by an (n,m)–torus bundle.

In [14], cusp cross-sections of real, complex, and quaternionic arithmetic hyperbolic n–orbifolds were classified. In this article, we continue this theme by classifying cusp cross-sections of Hilbert modular varieties. Before stating our first classification result, we introduce an additional piece of terminology.

For a totally real number field k, we say $\beta \in k$ is *totally positive* if $\sigma_j(\beta) > 0$ for $j = 1, \dots, n$. We denote the set of totally positive elements and totally positive integers by k_+ and $\mathcal{O}_{k,+}$, and define the sets $k_+^\times = k_+ \cap k^\times$, $\mathcal{O}_{k,+}^\times = \mathcal{O}_k^\times \cap \mathcal{O}_{k,+}$. We say that a virtual torus bundle N is *k–defined* if there exists a faithful representation $\rho\colon \pi_1(N) \longrightarrow k \rtimes k_+^\times$. If in addition $\rho(\pi_1(N))$ is commensurable with $\mathcal{O}_k \rtimes \mathcal{O}_{k,+}^\times$, we say that N is *k–arithmetic*.

Our first result is:

Theorem 1.1. *A virtual* $(n,n-1)$*–torus bundle N is diffeomorphic to a cusp cross-section of a Hilbert modular variety over k if and only if* $\pi_1(N)$ *is k–arithmetic.*

One direction of Theorem 1.1 is clear. The proof of the reverse implication makes use of techniques in [14] to lift an injective representation to be maximal peripheral. Theorem 1.1 answers a question of Hirzebruch [10, page 203] who implicitly asked (in our terminology) which k–arithmetic torus bundles arise as cusp cross-sections of Hilbert modular varieties. See Subsection 3.3 for more on this.

Every $(2,1)$–torus bundle admits either a Euclidean, Nil, or Sol geometry. Long and Reid [13] proved that the $(2,1)$–torus bundles which admit a Euclidean structure are diffeomorphic to cusp cross-sections of arithmetic real hyperbolic 4–orbifolds. In [14], we proved that those that admit Nil structures are diffeomorphic to cusp cross-sections of arithmetic complex hyperbolic 2–orbifolds. In this article, we prove (see §5 for the definitions):

Theorem 1.2. *Every* Sol *3–manifold is diffeomorphic to a cusp cross-section of a generalized Hilbert modular surface.*

As in [14], where we showed in higher dimensions there are always infranil manifolds which cannot be diffeomorphic to a cusp cross-section of any arithmetic complex hyperbolic n–orbifold, this smallest dimension is exceptional. Specifically, we can show that for $(n,n-1)$ with $n > 2$, there are $(n,n-1)$–torus bundles which admit a solvable structure and cannot be diffeomorphic to a cusp cross-section of any (generalized) Hilbert modular variety.

Using the Atiyah-Patodi-Singer signature formula [2, 3, 4], Long and Reid [12] showed that a flat 3–manifold which arises as a cusp cross-section of a 1–cusped real hyperbolic 4–manifold must have integral η–invariant. Together with Ouyang's work [18], this proves that certain flat 3–manifolds cannot be the cusp cross-section of a 1–cusped real hyperbolic 4–manifold. We conclude this article with a similar result. Specifically, using the work of Hirzebruch [10], Atiyah-Donnely-Singer [1], and Cheeger-Gromov [6], we prove:

Theorem 1.3. *There exists a* Sol *3–manifold which cannot be diffeomorphic to a cusp cross-section of any 1–cusped Hilbert modular surface with torsion free fundamental group.*

Acknowledgments

I would like to thank my advisor Alan Reid for all his help. In addition, I would like to thank Richard Schwartz for suggesting Hilbert modular varieties as a family of examples for which the techniques developed in [14] might be applied and for carefully reading an early draft of this article. Finally, I would like to thank John Hammond and Kate Petersen for helpful conversations on this material during its preparation.

2 Preliminary material

2.1 Stabilizer groups

We refer the reader to [20] for the basics of $\mathbf{H}^n_{\mathbf{R}}$ and its group of isometries.

For $v \in \partial \mathbf{H}^n_{\mathbf{R}}$, the group $\mathrm{Stab}(v) = \{\gamma \in \mathrm{Isom}(\mathbf{H}^n_{\mathbf{R}}) \ : \ \gamma v = v\}$ is isomorphic to $\mathbf{R}^{n-1} \rtimes (\mathbf{R}^+ \times \mathrm{O}(n-1))$. For $v \in \partial \mathbf{H}^n_{\mathbf{R}}$ and $H < \mathrm{Isom}(\mathbf{H}^n_{\mathbf{R}})$, we define the *stabilizer group of H at v* to be $\triangle_v(H) = \mathrm{Stab}(v) \cap H$. When $\triangle_v(H)$ contains a parabolic isometry, we call $\triangle_v(H)$ the *maximal peripheral subgroup of H at v* and say that H has a *cusp at v*. Often, we simply write $\triangle(H)$.

2.2 Infrasolv manifolds and smooth rigidity

For a simply connected, connected solvable Lie group S, the affine group of S is $\mathrm{Aff}(S) = S \rtimes \mathrm{Aut}(S)$. We say that a discrete subgroup $\Gamma < \mathrm{Aff}(S)$ is an *infrasolv group modelled on S* if $\Gamma \cap S$ is finite index in Γ and S/Γ is compact. An infrasolv group which is a subgroup of S will be called a *solv group modelled on S*. Any smooth manifold which is diffeomorphic to S/Γ for some infrasolv group will be called an *infrasolv manifold modelled on S*. When Γ is a solv group, we call the manifold S/Γ a *solv manifold modelled on S*.

We require the following rigidity result of Mostow [16].

Theorem 2.1 (Mostow; [16])**.** *Let M_1 and M_2 be infrasolv manifolds. If $\pi_1(M_1) \cong \pi_1(M_2)$, then M_1 is diffeomorphic to M_2.*

3 Cusps of Hilbert modular varieties

In this section, we prove Theorem 1.1. The philosophy for the proof is simple. Using the arithmeticity assumption on the torus bundle N, we construct an injective homomorphism $\rho\colon \pi_1(N) \longrightarrow \triangle(\mathrm{PSL}(2;\mathscr{O}_k))$. To find a Hilbert modular group Λ for which $\triangle(\Lambda) = \rho(\pi_1(N))$, we are reduced to making a subgroup separability argument. The proof is completed by applying Theorem 2.1. The remainder of this section is devoted to the details.

3.1 Subgroup separability

Recall that if G is a group, $H < G$ and $g \in G \setminus H$, we say H and g are *separated* if there exists a subgroup K of finite index in G which contains H but not g. We say that $H < G$ is *separable* in G if every $g \in G \setminus H$ and H can be separated.

As in [14], the main technical result we make use of is:

Theorem 3.1. *Let Λ be a Hilbert modular group and $\triangle(\Lambda)$, a maximal peripheral subgroup. Then every subgroup of $\triangle(\Lambda)$ is separable in Λ.*

3.2 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1. The following establishes a correspondence between k–arithmetic torus bundle groups and maximal peripheral subgroups of Hilbert modular groups.

Theorem 3.2 (Correspondence theorem)**.** *Let N be a k–arithmetic torus bundle. Then there exists a faithful representation $\psi\colon \pi_1(N) \longrightarrow \triangle(\mathrm{PSL}(2;\mathscr{O}_k))$ such that $\psi(\pi_1(N))$ is a finite index subgroup of $\triangle(\mathrm{PSL}(2;\mathscr{O}_k))$. Moreover, there exists a finite index subgroup Λ of $\mathrm{PSL}(2;\mathscr{O}_k)$ such that $\triangle(\Lambda) = \psi(\pi_1(N))$.*

We defer the proof of Theorem 3.2 for the moment in order to prove Theorem 1.1.

Proof of Theorem 1.1. For the direct implication, since N is diffeomorphic to a cusp cross-section of a Hilbert modular variety, there exists a Hilbert modular group Λ and an isomorphism $\psi\colon \pi_1(N) \longrightarrow \triangle(\Lambda)$. To obtain an injective homomorphism $\rho\colon \pi_1(N) \longrightarrow k \rtimes k_+^\times$ such that $\rho(\pi_1(N))$ is commensurable with $\mathscr{O}_k \rtimes \mathscr{O}_{k,+}^\times$, we argue as follows. By conjugating by an element γ of $\mathrm{PSL}(2;k)$, we can assume that

$$\gamma^{-1}\psi(\pi_1(N))\gamma \subset B_k = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} \ :\ \alpha \in k,\ \beta \in k_+^\times \right\}.$$

As $\gamma \in \mathrm{PSL}(2;k)$, $\gamma^{-1}\Lambda\gamma$ remains a Hilbert modular group, and moreover, $\gamma^{-1}\psi(\pi_1(N))\gamma$ is commensurable with

$$\triangle(\mathrm{PSL}(2;\mathscr{O}_k)) = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} \ :\ \alpha \in \mathscr{O}_k,\ \beta \in \mathscr{O}_{k,+}^\times \right\}.$$

To obtain the faithful representation ρ, we simply compose $\mu_\gamma \circ \psi$ with the isomorphism $\iota\colon B_k \longrightarrow k \rtimes k_+^\times$ given by $\iota\left(\begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix}\right) = (\alpha,\beta)$.

For the reverse implication, we apply Theorem 3.2 and Theorem 2.1. Specifically, let Λ by the Hilbert modular group guaranteed by Theorem 3.2 and let N' denote an embedded cusp cross-section associated with $\triangle(\Lambda)$. As a smooth manifold, N' is of the form $\mathbf{R}^{2n-1}/\triangle(\Lambda)$. By Theorem 3.2, we have an isomorphism $\psi\colon \pi_1(N) \longrightarrow \pi_1(N')$. Applying Theorem 2.1, we obtain the desired diffeomorphism between N and N'. $\square$

In the proof of Theorem 3.2, the following lemma is required.

Lemma 3.3. *Let N be a k–arithmetic torus bundle. Then there exists an injective homomorphism $\rho\colon \pi_1(N) \longrightarrow \mathscr{O}_k \rtimes \mathscr{O}_{k,+}^\times$. Moreover, $\rho(\pi_1(N))$ is a finite index subgroups of $\mathscr{O}_k \rtimes \mathscr{O}_{k,+}^\times$.*

Proof. Since N is k–arithmetic, we have a faithful representation $\theta\colon \pi_1(N) \longrightarrow k \rtimes k_+^\times$ such that $\theta(\pi_1(N))$ is commensurable with $\mathscr{O}_k \rtimes \mathscr{O}_{k,+}^\times$. Hence, given $(\alpha,\beta) \in \theta(\pi_1(N))$, we have for some $m \in \mathbf{N}$,

$$(\alpha + \beta\alpha + \beta^2\alpha + \cdots + \beta^{m-1}\alpha, \beta^m) \in \mathscr{O}_k \rtimes \mathscr{O}_{k,+}^\times.$$

Consequently, $\beta^m \in \mathscr{O}_{k,+}^\times$ and thus $\beta \in \mathscr{O}_{k,+}^\times$. Even so, it may be the case that (α,β) is not contained in $\mathscr{O}_k \rtimes \mathscr{O}_{k,+}^\times$. This is rectified as follows. Select a generating set for $\pi_1(N)$, say $g_1,\dots,g_u$. For each generator, we have $\theta(g_j) = (\alpha_j,\beta_j)$ with $\alpha_j \in k$

and $\beta_j \in \mathscr{O}_{k,+}^{\times}$. Since k is the field of fractions of $\mathscr{O}_k$, we can select $\lambda_j \in \mathscr{O}_k$ such that $(0,\lambda_j)\theta(g_j)(0,\lambda_j)^{-1} \in \mathscr{O}_k \rtimes \mathscr{O}_{k,+}^{\times}$. Note that

$$(0,\lambda_j)\theta(g_j)(0,\lambda_j)^{-1} = (\lambda_j\alpha_j,\beta_j),$$

and so the second coordinate β_j is unchanged. Finally, for $\lambda = \lambda_1 \dots \lambda_u$, define $\rho = \mu_{(0,\lambda)} \circ \theta$, where $\mu_{(0,\lambda)}$ denotes the inner automorphism determined by $(0,\lambda)$. By construction, ρ is a faithful representation of $\pi_1(N)$ onto a finite index subgroup of $\mathscr{O}_k \rtimes \mathscr{O}_{k,+}^{\times}$. □

With Lemma 3.3 in hand, we prove Theorem 3.2.

Proof of Theorem 3.2. By Lemma 3.3, we have an injective homomorphism $\rho : \pi_1(N) \longrightarrow \mathscr{O}_k \rtimes \mathscr{O}_{k,+}^{\times}$ such that $\rho(\pi_1(N))$ is a finite index subgroup. To obtain the injective homomorphism ψ, we compose ρ with the isomorphism

$$\iota^{-1} : \mathscr{O}_k \rtimes \mathscr{O}_{k,+}^{\times} \longrightarrow \triangle(\mathrm{PSL}(2;\mathscr{O}_k))$$

where $\iota^{-1}(\alpha,\beta) = \left(\begin{smallmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{smallmatrix}\right)$. That ψ is faithful and $\psi(\pi_1(N))$ is a finite index subgroup of $\triangle(\mathrm{PSL}(2;\mathscr{O}_k))$ follow immediately from the properties of ρ and ι.

To find the desired subgroup Λ, we apply Theorem 3.1. Specifically, select a complete set of coset representatives $\gamma_1,\dots,\gamma_s$ for $\triangle(\mathrm{PSL}(2;\mathscr{O}_k))/\psi(\pi_1(N))$. By Theorem 3.1, $\psi(\pi_1(N))$ is separable. Therefore for each j we can find finite index subgroups Λ_j such that $\gamma_j \notin \Lambda_j$ and $\psi(\pi_1(N)) < \Lambda_j$. To get the desired Λ, take $\Lambda = \bigcap_{j=1}^{s} \Lambda_j$. □

3.3 A question of Hirzebruch

Let k be a totally real number field, $M < k$ an additive group of rank n (the degree of k over $\mathbf{Q}$), and $V < \mathscr{O}_{k,+}^{\times}$ a finite index subgroup such that for all $\lambda \in V$, $\lambda M \subset M$. For each pair (M,V), we define the peripheral group

$$\triangle(M,V) = \left\{ \begin{pmatrix} \beta^{-1} & \alpha \\ 0 & \beta \end{pmatrix} \; : \; \alpha \in M,\ \beta \in V \right\} < \mathrm{PSL}(2;k).$$

For any Hilbert modular variety, the peripheral groups $\triangle(\Lambda)$ are conjugate (in $\mathrm{PSL}(2;k)$) to groups of the form $\triangle(M,V)$. In [10, p. 203], Hirzebruch mentions that it is apparently unknown whether or not every $\triangle(M,V)$ can occur as a maximal peripheral subgroup of a Hilbert modular group. The following corollary gives an affirmative answer.

Corollary 3.4. *For every pair (M,V), there exists a Hilbert modular group Λ such that $\triangle(\Lambda) = \triangle(M,V)$.*

Proof. As in the proof of Lemma 3.3, we can conjugate $\triangle(M,V)$ by an element of the form $\gamma = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$, with $\lambda \in \mathscr{O}_k$, such that $\gamma^{-1}\triangle(M,V)\gamma$ is contained in $\mathrm{PSL}(2;\mathscr{O}_k)$. Since M and V are finite index subgroups of $\mathscr{O}_k$ and $\mathscr{O}_{k,+}^{\times}$, respectively, $\gamma^{-1}\triangle(M,V)\gamma$ is a finite index subgroup of $\triangle(\mathrm{PSL}(2;\mathscr{O}_k))$. Thus there exists a finite index subgroup $\Lambda_1 < \mathrm{PSL}(2;\mathscr{O}_k)$ such that $\triangle(\Lambda_1) = \gamma^{-1}\triangle(M,V)\gamma$. Hence, for $\Lambda = \gamma\Lambda_1\gamma^{-1}$, we have $\triangle(\Lambda) = \triangle(M,V)$. As $\gamma \in \mathrm{PSL}(2;k)$, Λ is a Hilbert modular group, as required. $\square$

4 A simple criterion for arithmeticity

In this section, we give a simple criterion for the arithmeticity of (n,m)–torus bundles. The need for such a result is practical, as it allows one to establish the arithmeticity of a torus bundle computationally. We encourage the reader to compare the results of this section with Corollary 5.5 in [14].

4.1 Linear equations and presentations of torus bundle groups

For an (orientable) $(n,n-1)$–torus bundle M, since both the base and fiber are aspherical, we have the short exact sequence induced by the long exact sequence of the fiber bundle

$$1 \longrightarrow \mathbf{Z}^n \longrightarrow \pi_1(M) \longrightarrow \mathbf{Z}^{n-1} \longrightarrow 1.$$

The action of $\mathbf{Z}^{n-1}$ on $\mathbf{Z}^n$ induces a homomorphism $\varphi\colon \mathbf{Z}^{n-1} \longrightarrow \mathrm{SL}(n;\mathbf{Z})$ called the *holonomy representation*. Since peripheral subgroups in Hilbert modular groups have faithful holonomy representation, we assume throughout that φ is faithful. In particular, we obtain a faithful representation of $\pi_1(M)$ into $\mathbf{Z}^n \rtimes \mathrm{SL}(n;\mathbf{Z})$.

Of primary importance for us here is that the holonomy representation together with any finite presentation yields a homogenous linear system of equations with coefficients in $\mathbf{Z}$. This system arises as follows. For ease, select a presentation of the form

$$\langle x_1,\dots,x_n,\overline{y_1},\dots,\overline{y_{n-1}} \ : \ R\rangle$$

where $x_1,\dots,x_n$ generate $\mathbf{Z}^m$, $\overline{y_1},\dots,\overline{y_{n-1}}$ are lifts of a generating set $y_1,\dots,y_{n-1}$ for $\mathbf{Z}^{n-1}$, and R is a finite set of relations of the form

$$x_j\overline{y_k} = \overline{y_k}w_{j,k}, \quad w_{j,k} \in \langle x_1,\dots,x_n\rangle\,.$$

Using the holonomy representation, we can write

$$x_j = (a_j, I), \quad \overline{y_j} = (b_j, \varphi(y_j)) \in \mathbf{R}^n \rtimes \mathrm{SL}(n;\mathbf{R}).$$

Each relation in the presentation yields a linear homogenous equation in the vector variables a_j and b_j (see below for an explicit example of how these equations arise). Namely, we insert the above forms for x_j and $\overline{y_k}$ into the relation and consider only the first coordinate. The equations we obtain are of the form

$$a_j + b_k - \varphi(y_k) - v_{j,k} = 0$$

where $w_{j,k} = (v_{j,k}, I)$. That this system has integral solutions which yield faithful representations follows from the fact that φ is faithful and induces a faithful representation of $\pi_1(M)$ into $\mathbf{Z}^n \rtimes \mathrm{SL}(n;\mathbf{Z})$.

4.2 A simple criterion for arithmeticity

The main result of this section is a simple criterion for arithmeticity based on the structure of the holonomy representation. In the statement and proof, let $\mathrm{Res}_{k/\mathbf{Q}}$ denotes restriction of scalars from k to $\mathbf{Q}$ and assume that $[k:\mathbf{Q}] = n$ and $\mathrm{rank}\,\mathscr{O}_k^\times = n-1$. In particular, k is totally real.

Theorem 4.1. *Let M be an orientable $(n,n-1)$–torus bundle. Then M is diffeomorphic to a cusp cross-section of a Hilbert modular variety defined over k if and only if $\varphi = Res_{k/\mathbf{Q}}(\chi)$, for some faithful character $\chi\colon \mathbf{Z}^{n-1} \longrightarrow \mathscr{O}_{k,+}^\times$, where φ is some holonomy representation.*

Proof. For the direct implication, since M is diffeomorphic to a cusp cross-section of a Hilbert modular variety, by Theorem 1.1, we have a faithful representation

$$\rho\colon \pi_1(M) \longrightarrow \mathscr{O}_k \rtimes \mathscr{O}_{k,+}^\times$$

By restricting scalars from k to $\mathbf{Q}$, we obtain a faithful representation

$$\mathrm{Res}_{k/\mathbf{Q}}(\rho)\colon \pi_1(M) \longrightarrow \mathbf{Z}^n \rtimes \mathrm{SL}(n;\mathbf{Z}).$$

The proof is completed by noting that the holonomy map induced by this representation is simply $\mathrm{Res}_{k/\mathbf{Q}}(\chi)$, where $\chi\colon \mathbf{Z}^{n-1} \longrightarrow \mathscr{O}_{k,+}^\times$ is the holonomy representation induced by the representation ρ.

For the converse, we seek a faithful representation $\rho\colon \pi_1(M) \longrightarrow \mathscr{O}_k \rtimes \mathscr{O}_{k,+}^\times$. Note that since $[k:\mathbf{Q}] = n$ and $\mathrm{rank}\,\mathscr{O}_k^\times = n-1$, the image of $\pi_1(M)$ would necessarily be a finite index subgroup. By assumption, we have a faithful character

$\chi\colon \mathbf{Z}^{n-1} \longrightarrow \mathscr{O}_{k,+}^{\times}$. We extend this to a faithful representation of $\pi_1(M)$ into $\mathscr{O}_k \rtimes \mathscr{O}_{k,+}^{\times}$ as follows. Select a presentation as above for $\pi_1(M)$ with generators $x_1,\dots,x_n,\overline{y_1},\dots,\overline{y_{n-1}}$. Write

$$x_i = (\alpha_i, 1),\ \overline{y_i} = (\gamma_i, \chi(y_i)) \in k \rtimes \mathscr{O}_{k,+}^{\times} \tag{1}$$

where α_i and γ_i are to be determined. Using our presentation for $\pi_1(M)$, we obtain a system of linear homogenous equations $\mathscr{L}$ with coefficients in $\mathscr{O}_k$. Note, as above, solutions to $\mathscr{L}$ yield representations of $\pi_1(M)$ into $k \rtimes \mathscr{O}_{k,+}^{\times}$. We assert that there is a solution which yields a faithful representation. To see this, by restricting scalars from k to $\mathbf{Q}$, we obtain a linear system $\mathrm{Res}_{k/\mathbf{Q}}(\mathscr{L})$ with coefficients in $\mathbf{Z}$. Solutions to the system $\mathrm{Res}_{k/\mathbf{Q}}(\mathscr{L})$ yield representations of $\pi_1(M)$ into $\mathbf{Z}^n \rtimes \mathrm{SL}(n;\mathbf{Z})$. Moreover, a solution to $\mathrm{Res}_{k/\mathbf{Q}}(\mathscr{L})$ which yields a faithful representation is equivalent to a solution of $\mathscr{L}$ which yields a faithful representation into $\mathscr{O}_k \rtimes \mathscr{O}_{k,+}^{\times}$. That such a solution exists with integral coefficients for $\mathrm{Res}_{k/\mathbf{Q}}(\mathscr{L})$ follows from the faithfulness of $\mathrm{Res}_{k/\mathbf{Q}}(\chi)$ and our discussion in the previous subsection. This yields a solution for $\mathscr{L}$ with coefficients in $\mathscr{O}_k$ which yields a faithful representation. Therefore, M is k–arithmetic, since there exists a faithful representation $\psi\colon \pi_1(M) \longrightarrow \mathscr{O}_k \rtimes \mathscr{O}_{k,+}^{\times}$ such that $\psi(\pi_1(M))$ is a finite index subgroup of $\mathscr{O}_k \rtimes \mathscr{O}_k^{\times}$. □

Remark. If the character χ only maps into $\mathscr{O}_k^{\times}$, the above proof yields a faithful representation $\rho\colon \pi_1(M) \longrightarrow \mathscr{O}_k \rtimes \mathscr{O}_k^{\times}$.

5 Sol 3–manifolds

Before proving Theorem 1.2, we give a brief review of Sol 3–manifolds (see [22]). Let $\mathrm{Sol} = \mathbf{R}^2 \times \mathbf{R}^+$ with group operation defined by

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) \stackrel{\text{def}}{=} (x_1 + e^{t_1} x_2, y_1 + e^{-t_1} y_2, t_1 + t_2).$$

By a Sol *3–orbifold*, we mean a manifold M which is diffeomorphic to Sol/Γ, where Γ is a discrete subgroup of $\mathrm{Aff}(\mathrm{Sol})$ such that Sol/Γ is compact and $[\Gamma : \Gamma \cap \mathrm{Sol}] < \infty$. These manifolds, in the terminology from §2, are infrasolv manifolds modelled on Sol. However, the terminology used in this section for these manifolds is more prevalent (see [22] or [25]).

In [22], Scott proved that every $(2,1)$–torus bundles admits either a Euclidean, Nil, or Sol structure. The following result is easily derived from [22]. We include a proof here for completeness.

Proposition 5.1. *Let M be an orientable $(2,1)$–torus bundle which admits a* Sol *structure. Then there exists a faithful representation $\rho\colon \pi_1(M) \longrightarrow \mathscr{O}_k \rtimes \mathscr{O}_k^\times$ for some real quadratic number field k.*

Proof. For any $(2,1)$–torus bundle M, let the $\mathbf{Z}$–action be given by $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. If the order of A is finite, then $\pi_1(M)$ is a Bieberbach group and M admits a Euclidean structure. Therefore we may assume that the order of A is infinite. If A is not diagonalizable, then some power of A is conjugate to $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ with $\alpha \neq 0$. In this case, M admits a Nil structure. Thus, we may assume that A is diagonalizable. In this case we have $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ for a conjugate of A. It follows, since $A \in \mathrm{SL}(2;\mathbf{Z})$, that β and β^{-1} are algebraic integers in the real quadratic field $\mathbf{Q}(\beta)$. Thus the representation $\varphi\colon \mathbf{Z} \longrightarrow \mathrm{GL}(2;\mathbf{Z})$ is conjugate to $\mathrm{Res}_{k/\mathbf{Q}}(\chi)$, where $\chi\colon \mathbf{Z} \longrightarrow \mathscr{O}_k^\times$ is given by $\chi(1) = \beta$. Therefore by the remark following Theorem 4.1, we have a faithful representation $\rho\colon \pi_1(M) \longrightarrow \mathscr{O}_k \rtimes \mathscr{O}_k^\times$, as asserted. $\square$

Via Proposition 5.1, note every Sol 3–manifold group does faithfully represent into $\mathrm{Isom}((\mathbf{H}^2_\mathbf{R})^2)$. Those that arise as cusp cross-sections of Hilbert modular surfaces are precisely the ones whose fundamental group faithfully represents into the identity component of $\mathrm{Isom}((\mathbf{H}^2_\mathbf{R})^2)$. However, the quotients of those groups which fail to map into the identity component do produce finite volume quotients which possess 2–fold covers which are Hilbert modular surfaces. For this reason, we call such quotients *generalized Hilbert modular varieties*. Given this, Theorem 1.2 follows from this discussion in combination with Theorem 1.1.

6 Two examples

The following example illustrates the ideas in the proofs of Theorem 4.1 and Theorem 1.2.

Example 6.1. Let M be a $(2,1)$–torus bundle with $\mathbf{Z}$–action given by $A = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right)$. With this action, $\pi_1(M)$ has a presentation of the form (we are assuming that $\pi_1(M)$ is a split extension which is always the case; see [22])

$$\left\langle a_1, a_2, b \ : \ [a_1,a_2] = 1,\ ba_1 = a_1a_2b,\ ba_2 = a_1^2a_2^3b \right\rangle.$$

To obtain a faithful representation $\rho\colon \pi_1(M) \longrightarrow \mathscr{O}_k \rtimes \mathscr{O}_k^\times$ for some quadratic number field k, we first compute the eigenvalues of A. The characteristic polynomial for A is $p(t) = t^2 - 4t + 1$, which has roots $2 \pm \sqrt{3}$. Let $k = \mathbf{Q}(\sqrt{3})$ and write

$$\rho(a_1) = (x_1 + y_1\sqrt{3}, 1), \quad \rho(a_2) = (x_2 + y_2\sqrt{3}, 1), \quad \rho(b) = (0, 2 + \sqrt{3})$$

where x_1, x_2, x_3, and x_4 are to be determined. Using the presentation above, we are now reduced to solving a system of equations in x_1, x_2, x_3, and x_4 to ensure that ρ is an injective homomorphism. By construction $\rho([a_1, a_2]) = 1$. The other two relations yield the equations

$$(2+\sqrt{3})(x_1+y_1\sqrt{3}) = x_1+y_1\sqrt{3}+x_2+y_2\sqrt{3}$$
$$(2+\sqrt{3})(x_2+y_2\sqrt{3}) = 2(x_1+y_1\sqrt{3})+3(x_2+y_2\sqrt{3}).$$

Solving, we get the faithful representation

$$\rho(a_1) = (1,1), \quad \rho(a_2) = (1+\sqrt{3},1), \quad \rho(b) = (0,2+\sqrt{3}).$$

In fact such solutions abound. This is evident from the fact that a dense set of solvable structures on M arise as cusp cross-sections of Hilbert modular surfaces.

Let k be a totally real, cubic Galois extension of $\mathbf{Q}$ and $\beta \in \mathscr{O}_{k,+}^{\times}$ be of infinite order. By restricting scalars, we can view β as an element of $\mathrm{SL}(3;\mathbf{Z})$. Let M be a $(3,1)$–torus bundle with $\mathbf{Z}$–action given by β. Following the proof of Theorem 4.1, we can construct a faithful representation of $\pi_1(M)$ into $\mathscr{O}_k \rtimes \mathscr{O}_{k,+}^{\times}$. On the other hand, any Hilbert modular group defined over k cannot contain $\pi_1(M)$ as a finite index subgroup, since rank $\mathscr{O}_k^{\times} = 2$.

The following example is a specific case of the above.

Example 6.2. Define $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$ and let M be the $(3,1)$–torus bundle with $\mathbf{Z}$–action given by A. The characteristic polynomial for A is $c_A(t) = t^3+t^2-2t-1$. This polynomial is irreducible over $\mathbf{Q}$ and has totally real cubic splitting field. Thus, M is a $(3,1)$–torus bundle which is not diffeomorphic to a cusp cross-section of any generalized Hilbert modular variety. However, for the splitting field k of A, we do have an injection $\rho\colon \pi_1(M) \longrightarrow \triangle(\mathrm{PSL}(2;\mathscr{O}_k))$.

More generally, for any pair (n,m) with $n > 2$ and $m > 0$ we can construct an (n,m)–torus bundle which cannot be diffeomorphic to a cusp cross-section of any Hilbert-Blumenthal modular variety (see [15] and §8).

7 Geometric bounding

For a Hilbert modular surface W with torsion free $\pi_1(W)$, we call W a *Hilbert modular manifold*. In this case, W has finitely many cusp ends (or parabolic singularities). Following Long and Reid [12], we say that a Sol 3–manifold S *geometrically* *bounds* if it occurs as a cusp cross-section of a 1–cusped Hilbert modular manifold. Note that by neutering W (see [21] and [8]), we obtain a manifold $\widetilde{W}$ with $\partial\widetilde{W} = S$. Moreover, the locally symmetric metric $\widetilde{g}$ on W restricted to S endows S with a complete Sol metric g such that $\widetilde{g}$ is a complete, finite volume metric in the interior of $\widetilde{W}$ and (S,g) is a totally geodesic boundary.

For a closed $4n$–manifold M the Hirzebruch signature formula (see [11]) relates the signature of M with a certain polynomial (which depends only on n) evaluated on the Pontrjagin classes of M. In [10], Hirzebruch extended the signature formula to Hilbert modular surfaces. The formula relates the signature of the neutered manifold $\widetilde{W}$ again to a Hirzebruch L–polynomial evaluated on the Pontrjagin classes of $\widetilde{W}$ but with a correction term associated to $\partial\widetilde{W}$. When $\pi_1(W)$ is contains torsion the elliptic singularities also contribute nontrivially to this correction term. For simplicity, we assume throughout that $\pi_1(W)$ is torsion free. Then $\sigma(\widetilde{W}) = \delta(x_1)+\cdots+\delta(x_r)$ where $x_1,\dots,x_r$ is a complete set of parabolic singularities and $\sigma(\widetilde{W})$ denotes the signature of $\widetilde{W}$. The terms $\delta(x_j)$ are defined as follows. Each parabolic singularity is associated to a $\pi_1(W)$–conjugacy class Γ_j, where Γ_j is a maximal peripheral subgroup. The group Γ_j can be conjugated (in $\mathrm{PSL}(2;k)$) to a subgroup of the form $\triangle(M,V)$ (see §3.3). For the pair (M,V), we have the associated Shimuzu L–function $L(M,V,s)$ (see [23]) defined by

$$L(M,V,s) = \sum_{\beta\in(M\setminus\{0\})/V} \frac{\mathrm{sign}(N_{k/\mathbf{Q}}(\beta))}{(N_{k/\mathbf{Q}}(\beta))^s}$$

where $N_{k/\mathbf{Q}}$ is the norm map. The invariant $\delta(x_j)$ is then given by

$$\delta(x_j) = \frac{-\mathrm{vol}(M)}{\pi^2}L(M,V,1)$$

where $\mathrm{vol}(M)$ is the volume of $\mathbf{R}^2/M$ with respect to the pairing $\mathrm{Tr}_{k/\mathbf{Q}}$. Equivalently, $\mathrm{vol}(M) = \left|\det(\beta_i^{(j)})\right|$, where β_1,β_2 is a $\mathbf{Z}$–module basis for M and $\beta_i^{(1)}$ and $\beta_i^{(2)}$ denote the image of β_i under the two real embeddings of $k \longrightarrow \mathbf{R}$.

Theorem 7.1 (Hirzebruch;[10])**.** *Let W be a Hilbert modular manifold with exactly one cusp. Then*

$$\sigma(\widetilde{W}) = \frac{-\mathrm{vol}(M)}{\pi^2}L(M,V,1)$$

for the unique $\pi_1(W)$–conjugacy class $\triangle(M,V)$.

Associated to the $\mathbf{Z}$–module M is the so-called *dual lattice* M^* defined to be the image of M under the duality pairing provided by $\mathrm{Tr}_{k/\mathbf{Q}}$.

Proposition 7.2. *For a horosphere $\mathscr{H}$ stabilized by $\triangle(M,V)$ and $\triangle(M^*,V)$, $\mathscr{H}/\triangle(M,V)$ and $\mathscr{H}/\triangle(M^*,V)$ are diffeomorphic* Sol *3–manifolds.*

Proof. Let $\varphi_M, \varphi_{M^*}\colon V \longrightarrow \mathrm{SL}(2;\mathbf{Z})$ be the holonomy representations for $\triangle(M,V)$ and $\triangle(M^*,V)$. The pairing $\mathrm{Tr}_{k/\mathbf{Q}}$ can be viewed as an element of $\lambda \in \mathrm{SL}(2;\mathbf{Z})$ such that $\lambda M = M^*$. By construction $\varphi_{M^*} = \lambda(\varphi_M)\lambda^{-1}$. Thus, we have an isomorphism $\rho\colon \triangle(M,V) \longrightarrow \triangle(M^*,V)$ given by

$$\rho(\beta, \varphi_M(\alpha)) = (\lambda\beta, \lambda\varphi_M(\alpha)\lambda^{-1}).$$

The proof is completed by appealing to Theorem 2.1. □

Hecke [9] related the L–functions $L(M,V,s)$ and $L(M^*,V,s)$ by a functional equation. Specifically, for

$$H(M,V,s) = \left[\Gamma\left(\frac{s+1}{2}\right)\right]^2 \pi^{-(s+1)}\,[\mathrm{vol}(M)]^s\, L(M,V,s),$$

Hecke [9] proved that $H(M,V,s) = (-1)^s H(M^*,V,1-s)$. Thus for $s=1$ we obtain

$$(\Gamma(1))^2\,\pi^{-2}\,\mathrm{vol}(M)L(M,V,1) = -\left(\Gamma\left(\frac{1}{2}\right)\right)^2 \pi^{-1} L(M^*,V,0)$$
$$L(M^*,V,0) = -\frac{\mathrm{vol}(M)}{\pi^2} L(M,V,1),$$

and by Theorem 7.1, $\sigma(\widetilde{W}) = L(M^*,V,0)$. Indeed, this was shown in [1] (see also [17]) where $L(M^*,V,0)$ was reinterpreted as the η–invariant of an adiabatic limit. For this, we require some more terminology. As it is not required here, we do not define the η–invariant and refer the reader to [1].

Given a peripheral group $\triangle(M,V)$ and stabilized horosphere $\mathscr{H}$, the metric structure on the symmetric space $\mathbf{H}^2_{\mathbf{R}} \times \mathbf{H}^2_{\mathbf{R}}$ endows $\mathscr{H}$ with a metric structure which is visibly invariant under the action of $\triangle(M,V)$. Consequently the metric $g_{\mathscr{H},M,V}$ descends to quotient $\mathscr{H}/\triangle(M,V)$ and endows $\mathscr{H}/\triangle(M,V)$ with a complete Sol structure. In [1], the following was shown:

Theorem 7.3 (Atiyah-Donnely-Singer;[1])**.**

$$L(M^*,V,0) = \lim_{\varepsilon \longrightarrow 0} \eta(\mathscr{H}/\triangle(M^*,V), g_{\mathscr{H},M^*,V}/\varepsilon).$$

More generally, given any Sol structure g on S, we can define $\delta(S,g) = \lim_{\varepsilon\longrightarrow 0}\eta(S,g/\varepsilon)$. Critical in our proof of Theorem 1.3 is the independence of $\delta(S,g)$ from g shown by Cheeger and Gromov [6] (see [7] for a treatment specific to Sol). Specifically,

Theorem 7.4 (Cheeger-Gromov;[6]). *$\delta(S,g)$ is a topological invariant of the* Sol *3–manifold S.*

We are now in position to state and prove the principal observation needed in the proof of Theorem 1.3 (compare with [12]).

Theorem 7.5. *If S is diffeomorphic to a cusp cross-section of a 1–cusped Hilbert modular manifold, then $\delta(S) \in \mathbf{Z}$.*

Proof. Assume that (S,g) arises as a cusp cross-section of a 1–cusped Hilbert modular manifold W. Then there is an isometric embedding $f\colon (S,g) \longrightarrow W$ onto a cusp cross-section of W. Let $f_*(\pi_1(S)) = \triangle(M,V)$ with associated horosphere $\mathscr{H}$ selected such that $\mathscr{H}/\triangle(M,V)$ is embedded in W. By Proposition 7.2, $\mathscr{H}/\triangle(M^*,V)$ is diffeomorphic to S, though equipped with the metric $g_{\mathscr{H},M^*,V}$. From the computation above in combination with Theorem 7.3, $\sigma(\widetilde{W}) = \delta(S, g_{\mathscr{H},M^*,V})$ and by Theorem 7.4, the right hand side depends only on the topological type of S. Since $\sigma(\widetilde{W}) \in \mathbf{Z}$, $\delta(S) \in \mathbf{Z}$ as asserted. □

With Theorem 7.5 in hand, we now prove Theorem 1.3.

Proof of Theorem 1.3. To prove Theorem 1.3, by Theorem 7.5, it suffices to find a Sol 3–manifold S for which $\delta(S) \notin \mathbf{Z}$. For $k = \mathbf{Q}(\sqrt{3})$, the standard Hilbert modular surface W over k has precisely one cusp, since the number of cusps of a standard Hilbert modular surface is precisely the ideal class number of the number field. Setting S to be an embedding cusp cross-section of W, the proof is completed by appealing to [10]. Specifically, Hirzebruch showed $\delta(S) = -1/3$. □

Remark. The number fields $\mathbf{Q}(\sqrt{6})$, $\mathbf{Q}(\sqrt{21})$ and $\mathbf{Q}(\sqrt{33})$ also have standard Hilbert modular surfaces with precisely one cusp for which the associated invariant $\delta(S) \notin \mathbf{Z}$. In each of these cases, $\delta(S) = -2/3$ (see [10, p. 236]).

It is unknown to the author whether or not there exist 1–cusped Hilbert modular manifolds. Using the generalized Riemann hypothesis, K. Petersen [19] constructed infinite many 1–cusped Hilbert modular surfaces. However, the nature of the construction likely produces Hilbert modular surface groups with 2–torsion.

8 Final remarks and generalizations

As it requires no more work, we mention the case when k is any algebraic number field. Our interest is in subgroups of $\mathrm{PSL}(2;k)$ which are commensurable with $\mathrm{PSL}(2;\mathscr{O}_k)$. As above for Hilbert modular groups, these groups are arithmetic subgroups of $(\mathrm{PSL}(2;\mathbf{R}))^{r_1} \times (\mathrm{PSL}(2;\mathbf{C}))^{r_2}$, where r_1 is the number of real embeddings of k and r_2 is the number of complex embeddings of k, up to conjugation. These groups act on $(\mathbf{H}^2_{\mathbf{R}})^{r_1} \times (\mathbf{H}^3_{\mathbf{R}})^{r_2}$ and yield finite volume quotients. We call any subgroup of $\mathrm{PSL}(2;k)$ which is commensurable with $\mathrm{PSL}(2;\mathscr{O}_k)$ a *Hilbert-Blumenthal modular group* and call the quotients *Hilbert-Blumenthal modular varieties*.

The cusps of these quotients are (n,m)–torus bundles, where $n = [k : \mathbf{Q}]$ and $m = \operatorname{rank} \mathscr{O}_k^\times$. Before stating our next result, we require some terminology. Let $\sigma_1, \dots, \sigma_{r_1}$ and $\tau_1, \dots, \tau_{r_2}$ denote the real and complex embeddings of k, respectively. For each embedding, define $\widetilde{\sigma}_j \colon k^\times \longrightarrow \mathbf{R}^\times / \langle \pm 1 \rangle$ and $\widetilde{\tau}_j \colon k^\times \longrightarrow \mathbf{C}^\times / \langle \pm 1 \rangle$. The product of these maps yields

$$\rho \colon k^\times \longrightarrow \prod_{j=1}^{r_1} \mathbf{R}^\times / \langle \pm 1 \rangle \times \prod_{j=1}^{r_2} \mathbf{C}^\times / \langle \pm 1 \rangle \,.$$

We say that $V \subset k^\times$ is *positive* if $\rho_{|V}$ is an injective homomorphism. Note when k is totally real such a V consists of totally positive numbers.

We say that an (n,m)–torus bundle is *k–defined* if there exists a positive subgroup V in $k^\times$ and a faithful representation of $\pi_1(N)$ into $k \rtimes V$. If in addition, $\pi_1(N)$ is commensurable with $\mathscr{O}_k \rtimes \mathscr{O}_k^\times$ under this representation, we say that N is *k–arithmetic*.

Theorem 8.1. *A virtual (n,m)–torus bundle N is diffeomorphic to a cusp cross-section of a Hilbert-Blumenthal modular variety defined over k if and only if N is k–arithmetic.*

The proof of Theorem 8.1 is identical to Theorem 1.1, as is Theorem 4.1 in this more general setting. As before, for any pair (n,m) with $n > 2$ and $m > 0$ we can construct an (n,m)–torus bundle which cannot be diffeomorphic to a cusp cross-section of any Hilbert-Blumenthal modular variety (see [15]).

Using techniques identical to those in [14], we can classify cusp cross-sections of irreducible (noncocompact) lattices in the n–fold product $(n > 1)$

$$\mathbf{G}_{m_1,\dots,m_n} = \prod_{j=1}^{n} \mathrm{Isom}(\mathbf{H}_X^{m_j})$$

where $X = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. Irreducible lattices in $\mathbf{G}_{m_1,\dots,m_n}$ exist if and only if each $m_j = 2$ or 3 and $X = \mathbf{R}$, or $m_j = m_k$ for all j and k.

References

[1] M. F. Atiyah, H. Donnelly, and I. M. Singer, *Eta invariants, signature defects of cusps, and values of L–functions*, Ann. of Math. (2) **118** (1983), no. 1, 131–177.

[2] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

[3] ———, *Spectral asymmetry and Riemannian geometry. II*, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 3, 405–432.

[4] ———, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc. **79** (1976), no. 1, 71–99.

[5] Armand Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962), 485–535.

[6] Jeff Cheeger and Mikhael Gromov, *Bounds on the von Neumann dimension of L^2–cohomology and the Gauss-Bonnet theorem for open manifolds*, J. Differential Geom. **21** (1985), no. 1, 1–34.

[7] Jeff Cheeger, *η–invariants, the adiabatic approximation and conical singularities. I. The adiabatic approximation*, J. Differential Geom. **26** (1987), no. 1, 175–221.

[8] Benson Farb and Richard Schwartz, *The large-scale geometry of hilbert modular groups*, J. Differential Geom. **44** (1996), no. 3, 435–478.

[9] Erich Hecke, *Mathematische Werke*, third ed., Vandenhoeck & Ruprecht, Göttingen, 1983, With introductory material by B. Schoeneberg, C. L. Siegel and J. Nielsen.

[10] Friedrich E. P. Hirzebruch, *Hilbert modular surfaces*, Enseignement Math. (2) **19** (1973), 183–281.

[11] Friedrich Hirzebruch, *Topological methods in algebraic geometry*, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[12] D. D. Long and A. W. Reid, *On the geometric boundaries of hyperbolic* 4–*manifolds*, Geom. Topol. **4** (2000), 171–178 (electronic).

[13] ———, *All flat manifolds are cusps of hyperbolic orbifolds*, Algebr. Geom. Topol. **2** (2002), 285–296 (electronic).

[14] D. B. McReynolds, *Peripheral separability and cusps of arithmetic hyperbolic orbifolds*, Algebr. Geom. Topol. **4** (2004), 721–755 (electronic).

[15] ———, *Ph.d. thesis*, In preparation.

[16] G. D. Mostow, *Factor spaces of solvable groups*, Ann. of Math. (2) **60** (1954), 1–27.

[17] Werner Müller, *Signature defects of cusps of Hilbert modular varieties and values of L–series at* $s = 1$, J. Differential Geom. **20** (1984), no. 1, 55–119.

[18] Ming Qing Ouyang, *Geometric invariants for Seifert fibred* 3*–manifolds*, Trans. Amer. Math. Soc. **346** (1994), no. 2, 641–659.

[19] K. Petersen, *One-cusped congruence subgroups of* $\mathrm{PSL}(2;\mathcal{O}_k)$, Ph.d. Thesis, University of Texas (2005).

[20] John G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics, vol. 149, Springer-Verlag, New York, 1994.

[21] Richard Evan Schwartz, *The quasi-isometry classification of rank one lattices*, Inst. Hautes Études Sci. Publ. Math. (1995), no. 82, 133–168 (1996).

[22] Peter Scott, *The geometries of* 3*–manifolds*, Bull. London Math. Soc. **15** (1983), no. 5, 401–487.

[23] Hideo Shimizu, *On discontinuous groups operating on the product of the upper half planes*, Ann. of Math. (2) **77** (1963), 33–71.

[24] Norman Steenrod, *The topology of fibre bundles*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999.

[25] William P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997, Edited by Silvio Levy.

[26] Gerard van der Geer, *Hilbert modular surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 16, Springer-Verlag, Berlin, 1988.

Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, TX 78712-0257
email: `dmcreyn@math.utexas.edu`