Constructing isospectral manifolds

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Abstract

In this article we construct nonisometric, isospectral manifolds modelled on semisimple Lie groups with finite center and no compact factors. Specifically, our two main results are the construction of arbitrarily large sets of closed, isospectral, nonisometric manifolds and pairs of infinite towers of finite covers that are isospectral and nonisometric at each stage. We also show the growth of these large sets of isospectral manifolds as a function of their volume is super-polynomial.

1 Introduction and results

For a closed Riemannian n–manifold M^n and Laplace–Beltrami operator Δ, the *spectrum* of Δ acting on the Hilbert space $L^2(M)$ is discrete with each eigenvalue occurring with finite multiplicity. We denote the spectrum of Δ on $L^2(M)$ with multiplicities by $\mathcal{E}(M,\Delta)$ and say two closed Riemannian n–manifolds M_1 and M_2 are *isospectral* if $\mathcal{E}(M_1,\Delta) = \mathcal{E}(M_2,\Delta)$.

There is a long history on the construction of isospectral nonisometric manifolds and for brevity we only touch on those results most pertinent to this paper (see [11] for a survey). Sunada [39] produced one of the first general methods for constructing isospectral manifolds, and with this method, constructed isospectral, nonisometric surfaces. Using Sunada's method and strong approximation in simple groups, Spatzier [37] showed every compact, irreducible, locally symmetric manifold admits a pair of isospectral, nonisometric finite covers provided the universal cover's isometry group has a complexification of sufficiently high rank. Brooks, Gornet, and Gustafson [7], also using Sunada's method, constructed arbitrarily large sets of isospectral, nonisometric, nonarithmetic closed hyperbolic surfaces.

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1.1 Large sets of isospectral manifolds Our first result shows the existence of large sets of isospectral, nonisometric manifolds for general noncompact symmetric spaces—see Theorem 8.2 for our most general result.

Theorem 1.1. *If X is a symmetric space whose isometry group has finite center and no compact factors, then for every $n \in \mathbf{N}$, there exist n closed, isospectral, nonisometric manifolds whose universal cover is X.*

The following corollary was the primary motivation for our investigation.

Corollary 1.2. *If X is a symmetric space whose isometry group has finite center and no compact factors, then there exist closed, isospectral, nonisometric manifolds whose universal cover is X.*

Two notable classical geometries for which Corollary 1.2 is new are complex and quaternionic hyperbolic n–space. In the case of complex hyperbolic 2–manifolds, a simple proof is given in section 6 using a special commensurable class of complex hyperbolic 2–manifolds. However, for many manifolds, this simple approach cannot be carried out and Theorem 1.1 provides arbitrarily large collections of isospectral covers when even a single pair was not known to exist. In complex dimension 2 to 24, there were no previous examples, and in complex dimensions greater than 24, only Spatzier's examples were known. For quaternionic hyperbolic n–manifolds, only Spatzier's example were known and these arise in quaternionic dimension greater than 25.

In every setting Theorem 1.1 provides new examples. Indeed, our methods show that a generic manifold modelled on X as in Theorem 1.1 is commensurable with one possessing many isospectral, nonisometric finite covers. Even for hyperbolic n–manifolds where a considerable amount of energy has been devoted to the construction of isospectral manifolds ([4], [9], [35], [21], [41]), this provides many new examples—see section 8 and Corollary 8.3 for more on this.

For every exceptional geometry, both Theorem 1.1 and Corollary 1.2 are new. One example are manifolds modelled on the Cayley hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$ whose associated Lie group is the real quasi-split form $F_{4(-20)}$ of the exceptional complex simple Lie group F_4. Theorem 1.1 and Corollary 1.2 produce the first examples of isospectral, nonisometric 16–manifolds modelled on the Cayley hyperbolic plane.

1.2 Isospectral growth For a symmetric space X as in Corollary 1.2 and any real positive t, let $\mathrm{SD}_X(t)$ denote the cardinality of the largest set of isometry classes of closed X–manifolds that are pairwise isospectral and have volume no more than t. According to work of Pesce [28], $\mathrm{SD}_X(t)$ is finite for all t, and there are explicit upper bounds from counting methods (see [20]). Our next result produces the first general, nontrivial lower bounds for $\mathrm{SD}_X(t)$.

Theorem 1.3. *There exist increasing sequences t_j and r_j such that*

$$SD_X(t_j) \geq t_j^{r_j}.$$

In particular, $SD_X(t)$ is super-polynomial as a function of t.

When $X = \mathbf{H}^2_{\mathbf{R}}$, this result was established by Brooks–Gornet–Gustafson [7]—see Section 9 for a more detailed account of [7]. In most settings, this function was not previously known to be unbounded. Additionally, when $\mathrm{Isom}(X)$ possesses a nonarithmetic, large cocompact lattice, these bounds can be improved.

Theorem 1.4. *Let X be complex hyperbolic 2–space or real hyperbolic n–space. Then there exists a constant D and an increasing sequence t_j such that*

$$SD_X(t_j) \geq t_j^{D\log(t_j)}.$$

One can take for D the number $\frac{1}{8\log 2}$. We also mention that neither of these results achieve the same asymptotic growth as the number of X–manifolds of volume bounded by t.

1.3 Isospectral towers Given two manifolds M and N, we say the pair (M,N) possesses an *isospectral tower* if there exist two infinite towers of finite covers $\{M_j\}$ and $\{N_j\}$ of M,N such that for every j, M_j and N_j are isospectral and nonisometric. A manifold M is said to possess an *isospectral tower* if (M,M) possesses an isospectral tower.

Theorem 1.5. *If X is a symmetric space whose isometry group has finite center and no compact factors, then there exist closed manifolds whose universal cover is X which possess an isospectral tower.*

To our knowledge, the only other tower constructions are given by Vigérnas [41] for certain closed manifolds modelled on $\mathrm{SL}(2;\mathbf{R})$, $\mathrm{SL}(2;\mathbf{C})$, and products of these groups, and by Lubotzky–Samuel–Vishne [19] who constructed isospectral towers for certain pairs of closed locally symmetric manifolds modelled on $\mathrm{SL}(n;\mathbf{R})$ and $\mathrm{SL}(n;\mathbf{C})$ for $n > 2$. Even more remarkable is the fact that the towers constructed in [19] are for incommensurable manifolds M,N—compare with [35]—a feature which can never be achieved with Sunada's method.

1.4 The isometry problem and probabilistic methods The difficulty in [7] and [37] is ensuring that the isospectral manifolds built by the construction are nonisometric. In [37] (see also [38]), a direct argument was given while in [7] nonarithmetic hyperbolic surfaces provide strong bounds on the number of ways two manifolds can be isometric. With our construction and the typical settings we seek to apply it, neither of these approaches can be implemented. The isometry problem is overcome with recent work of Belolipetsky and Lubotzky [3], and not unlike the constructions of [2], our constructions are probabilistic—see section 4.

1.5 Applications Aside from the construction of isospectral manifolds, we have applications similar to those in [7] and [37]. The first is the number theoretic version of Theorem 1.5 (compare with [7])—see section 11 for the definitions and Theorem 12.2 for the most general result.

Theorem 1.6. *There exists two towers of finite extensions k_j, k_j' of $\mathbf{Q}$ such that k_j, k_j' are non-isomorphic and*

$$\zeta_{k_j}(s) = \zeta_{k_j'}(s)$$

for all $j \in \mathbf{N}$.

Our second application follows [37] who obtained this result for many isotypic semisimple Lie groups with classical factors.

Corollary 1.7. *If G is an isotypic semisimple Lie group with finite center and no compact factors, then there exist measurably distinct properly ergodic actions of G with equal, discrete spectra.*

With the validity of Corollary 1.2, the proof is identical to the one given in [37] and hence has been omitted. For many classical geometries and all exceptional geometries, this is new. In total, these results show the rank assumptions of [37] are not necessary, confirming a statement made by Spatzier in [37]. In addition, as these actions arise from pairs of isospectral, nonisometric manifolds modelled on G, we obtain arbitrarily large collections and pairs of towers of measurably distinct properly ergodic actions of G with equal, discrete spectra from Theorem 1.1 and Theorem 1.5.

Article layout In the second section, we introduce some notation and terminology to be used throughout the remainder of the article. In the third section, we discuss the isometry problem for covers and its relationship with the commensurator of a lattice. In the fourth section, we review Sunada's method and introduce discrete isospectral deformation. In the fifth section, we introduce the primary discrete isospectral deformation families derived from Heisenberg groups. Section six serves as an example of the methods of this article. Here we construct isospectral, nonisometric covers for a special class of nonarithmetic manifolds following [7]. In section seven, we produce discrete isospectral deformations for arithmetic lattices in semisimple Lie groups with finite center and no compact factors. In section eight, we prove Theorem 1.1, and in section nine we prove Theorem 1.3 and Theorem 1.4. In section ten, we prove Theorem 1.5. Finally, in section twelve we prove Theorem 1.6 whose proof uses the deformation families considered in the manifold constructions.

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2 Preliminary material

We refer the reader to [14] and [31] for a thorough discussion of the topics of this section.

For a number field F, we denote its ring of integers by $\mathcal{O}_F$, the set prime ideals of $\mathcal{O}_F$ by $\mathcal{P}(F)$, the equivalence classes of valuations on F by $V(F)$, and the subset of nonarchimedean valuations by $V_f(F)$. We denote the completion of F with respect to the valuation ν by F_ν, the adèles over F by $\mathbb{A}_F$, and the finite adèles by $\mathbb{A}_F^f$. For a subset S of $V(F)$, the V_S–adèles are defined to be the restricted product of F_ν over all $\nu \in V(F) \setminus S$ and shall be denoted by $\mathbb{A}_F(S)$.

For an F–algebraic group $\mathbf{G}$ and subring R of $\overline{F}$ containing $\mathcal{O}_F$, we denote the group of R–points by $\mathbf{G}(R)$, which is well defined up to commensurability. By a *real algebraic group*, we mean the real points of an algebraic group. For any ideal $\mathfrak{p}$ of R, $r_\mathfrak{p}$ will denote the reduction map modulo $\mathfrak{p}$, and we call the kernel of $r_\mathfrak{p}$ a *principal congruence subgroup* of $\mathbf{G}(R)$ and any subgroup of $\mathbf{G}(R)$ containing a principal congruence subgroup a *congruence subgroup*. We denote the *corestriction* or *restriction of scalars* functor from a number field L to a subfield k by $\mathrm{Res}_{L/k}$ and recall this is a functor from the category of L–algebraic groups to the category of k–algebraic groups.

By a *real noncompact classical group*, we mean the classical groups $\mathrm{SL}(n;\mathbf{R})$, $\mathrm{SO}(r,s)$, $\mathrm{SU}(r,s)$, and $\mathrm{Sp}(r,s)$ with $r \geq s \geq 1$ and $n > 1$. Due to exceptional isogenies, we exclude the groups $\mathrm{SL}(2;\mathbf{R})$, $\mathrm{SO}(2,2)$, $\mathrm{Sp}(1,1)$, and $\mathrm{SU}(1,1)$. Below, let $X = \mathbf{R}$, $\mathbf{C}$, or $\mathbb{H}$,

$$I_{r,s} = \begin{pmatrix} I_r & 0_s \\ -I_s & 0_r \end{pmatrix}, \quad J_n = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

and $*$ denote the conjugate transpose on $\mathrm{M}(n;X)$. Explicitly, the classical groups are defined below:

$$\begin{aligned}
\mathrm{SL}(n;X) &= \{A \in \mathrm{M}(n;X) \ : \ \det(A) = 1\} \\
\mathrm{SO}(r,s) &= \{A \in \mathrm{SL}(r+s;\mathbf{R}) \ : \ I_{r,s}A^*I_{r,s}A = I_{r+s}\} \\
\mathrm{SU}(r,s) &= \{A \in \mathrm{SL}(r+s;\mathbf{C}) \ : \ I_{r,s}A^*I_{r,s}A = I_{r+s}\} \\
\mathrm{Sp}(r,s) &= \{A \in \mathrm{SL}(r+s;\mathbb{H}) \ : \ I_{r,s}A^*I_{r,s}A = I_{r+s}\} \\
\mathrm{Sp}(2n;\mathbf{R}) &= \left\{A \in \mathrm{SL}(2n;\mathbf{R}) \ : \ J_n^{-1}A^*J_nA = I_{2n}\right\}.
\end{aligned}$$

2.1 An *isogeny* between two Lie groups G_1 and G_2 is a surjective Lie homomorphism $\rho\colon G_1 \longrightarrow G_2$ with finite kernel. A Lie group G is called *simply connected* if any isogeny $\rho\colon \widehat{G} \longrightarrow G$ with $\widehat{G}$ connected is an isomorphism and *adjoint* if any isogeny $\rho\colon G \longrightarrow \widehat{G}$ with $\widehat{G}$ connected is an isomorphism. It is a fundamental fact that for any semisimple Lie group G, there exists a unique simply connected $\widetilde{G}$ and adjoint group $\overline{G}$ in the isogeny class of G. In the case G is F–algebraic, both $\widetilde{G}$ and $\overline{G}$ are F–algebraic and the isogenies are F–defined.

Two Lie groups G_1, G_2 are *isogenous* if there exists a Lie group G_3 and a pair of isogenies

$$\begin{array}{ccc} G_1 & & G_2 \\ & \searrow \quad \swarrow & \\ & G_3 & \end{array}$$

This produces an equivalence relation and we refer to an equivalence class as an *isogeny class*.

2.2 For a semisimple Lie group G with maximal compact subgroup K, G/K is a symmetric Riemannian manifold, where the metric is induced by the selection of a bi-invariant Haar measure. When G is noncompact, G/K is said to be of *noncompact type*. It is a fundamental result of Cartan that every noncompact symmetric Riemannian manifold arises from such a construction. The discrete torsion free subgroups Γ of $\mathrm{Isom}(G/K)$ yield complete locally symmetric Riemannian manifolds $\Gamma\backslash G/K$, and in this case we say $\Gamma\backslash G/K$ is *modelled on* G. If we do not insist Γ be torsion free, $\Gamma\backslash G/K$ is a good orbifold in the sense of Thurston. When G has a finite center and no compact factors, the isometry group $\mathrm{Isom}(G/K)$ and G are isogenous and $\mathrm{Isom}(G/K)_0$ is the adjoint form of G.

2.3 The following theorem is due to Nori [26] and Weisfeiler [43], and is known as the *Strong Approximation Theorem* (see also Pink [30]).

Theorem (Strong Approximation)**.** *If* $\mathbf{G}$ *is an absolutely simple, connected, simply connected k–algebraic group, R a finitely generated subring of k whose field of fractions is k, and* Γ *a Zariski dense subgroup of* $\mathbf{G}(R)$ *whose ring of traces is R, then the closure of* Γ *in* $\mathbf{G}(\mathbb{A}(k))$ *is open and compact.*

3 Commensurators and lattices in the classical groups

For a lattice Λ, if $\Lambda\backslash G$ is compact, we say Λ is *cocompact* (or *uniform*) and *noncocompact* (or *nonuniform*) otherwise. It is a fundamental result of Borel and Harish-Chandra [6] that $\mathbf{G}(\mathbf{Z})$ is a lattice in $\mathbf{G}$ when $\mathbf{G}$ is a real semisimple $\mathbf{Q}$–algebraic group. More generally, for a real Lie group G with discrete subgroup Λ, we say Λ is an *arithmetic lattice* if there exists a real $\mathbf{Q}$–algebraic group $\mathbf{G}$, compact Lie groups K_1 and K_2, and an exact sequence

$$1 \longrightarrow K_1 \longrightarrow \mathbf{G} \xrightarrow{\psi} G \longrightarrow K_2 \longrightarrow 1$$

such that $\psi(\mathbf{G}(\mathbf{Z}))$ is commensurable with Λ. For semisimple Lie groups G, one can take $K_2 = 1$ (see [24]).

3.1 By a *real k–form* of a Lie group G we mean a real k–algebraic group $\mathbf{H}$ together with a Lie epimorphism $\rho\colon \mathbf{H} \longrightarrow G$ with compact kernel. We say $\mathbf{H}$ is *admissible* if $\mathbf{H}(\mathcal{O}_k)$ is a lattice in $\mathbf{H}$. In this case, we call $\rho(\mathbf{H}(\mathcal{O}_k))$ a *principal arithmetic subgroup* of G.

In [40], Tits classified the real k–forms of the real simple Lie groups. In this collection, the admissible k–forms are precisely those for which the kernel of the projection $\mathrm{Res}_{k/\mathbf{Q}}(\mathbf{H}) \longrightarrow \mathbf{H}$ is compact.

3.2 For a lattice Λ of G, the *commensurator of* Λ is the subgroup of elements γ in G such that $\gamma^{-1}\Lambda\gamma$ and Λ are commensurable and is denoted by $\mathrm{Comm}_G(\Lambda)$. The following result of Margulis is sometimes referred to as *Margulis' dichotomy*—a proof can be found in [22, Thm. 1] or [44, Thm. 6.2.5].

Theorem 3.1. *If G is a semisimple Lie group with finite center and no compact factors, and Λ an irreducible lattice in G, then Λ is arithmetic if and only if $[\mathrm{Comm}_G(\Lambda):\Lambda]$ is infinite.*

The commensurator of a lattice Λ admits a natural action on the commensurability class of Λ given by

$$\alpha(\Lambda') = \alpha^{-1}\Lambda'\alpha.$$

For a finite index subgroup $\triangle$ of Λ, we denote by $\mathrm{Orbit}(\triangle)$, the commensurator orbit of $\triangle$ intersected with the set $\mathcal{FI}(\Lambda)$ of finite index subgroups of Λ. For any positive integer t, $\mathrm{Orbit}_t(\triangle)$ will denote the intersection of the commensurator orbit of $\triangle$ with the set $\mathcal{FI}_t(\Lambda)$ of index t subgroups of Λ.

For an adjoint group G not isogenous to $\mathrm{PSO}(2,1)$, by strong rigidity (see [25], [33], and [22, Ch. VII]), two lattices Λ_1,Λ_2 are isomorphic if and only if Λ_1 and Λ_2 are conjugate in $\mathrm{Isom}(G/K)$. In particular, the orbit $\mathrm{Orbit}_{[\Lambda:\triangle]}(\triangle)$ under the $\mathrm{Comm}_{\mathrm{Isom}(G/K)}(\Lambda)$–action intersected with $\mathcal{FI}(\Lambda)$ parameterizes the $[\Lambda:\triangle]$–sheeted covers of $\Lambda\backslash G/K$ that are isometric to $\triangle\backslash G/K$. The following is an immediate consequence of Margulis' dichotomy.

Lemma 3.2. *Let Λ be a nonarithmetic lattice and $\triangle$ is a finite index subgroup of Λ. Then the number distinct Λ–conjugacy classes in* $\mathrm{Orbit}_{[\Lambda:\triangle]}(\triangle)$ *is bounded above by* $[\mathrm{Comm}_{\mathrm{Isom}(G/K)}(\Lambda):\triangle]$.

When applying this result, to avoid a cumbersome reference to $\mathrm{Comm}_{\mathrm{Isom}(G/K)}(\Lambda)$, we simply note the number of Λ–distinct conjugacy classes in $\mathrm{Orbit}_{[\Lambda:\triangle]}(\Lambda)$ is bounded above by

$$C'[\mathrm{Comm}_G(\Lambda):\Lambda],$$

where C' is a constant depending only on Λ.

For arithmetic lattices, controlling the size of a commensurator orbit for a fixed finite index subgroup is difficult. For principal arithmetic groups and their congruence subgroups, Belolipetsky and Lubotzky [3] achieved such a result.

Theorem 3.3 (Belolipetsky–Lubotzky)**.** *If* $\mathbf{G}$ *is a connected, simply connected, absolutely simple k–algebraic group not isogenous to* $\mathrm{SO}_0(2,1)$, *then for any principal arithmetic subgroup Λ of* $\mathbf{G}$ *and any congruence subgroup $\triangle$ of Λ,*

$$\left|\mathrm{Orbit}_{[\Lambda:\triangle]}(\triangle)\right| \le [\Lambda:\triangle]\,|\mathcal{O}_k/\mathfrak{a}|^{\dim \mathbf{G}}$$

where $\triangle_0$ *is a principal congruence subgroup for the ideal* $\mathfrak{a}$ *contained in* $\triangle$ *and* $\dim \mathbf{G}$ *is the real dimension of* $\mathbf{G}$.

If $\mathbf{G}$ is a real k–algebraic subgroup of $\mathrm{GL}(m;\mathbf{C})$, $\Lambda = \mathbf{G}(\mathcal{O}_k)$, and $\triangle_0$ is the kernel of the reduction map $r_{\mathfrak{a}}$ for an ideal $\mathfrak{a}$ of $\mathcal{O}_k$, the above bound for any $\triangle$ in Λ containing $\triangle_0$ is

$$\left|\mathrm{Orbit}_{[\Lambda:\triangle]}(\triangle)\right| \leq |\mathcal{O}_k/\mathfrak{a}|^{(\dim \mathbf{G})^2}.$$

It is this form of Theorem 3.3 which we shall apply in the sequel. Note, if our concern lies with distinct Λ–conjugacy classes, there are at most

$$|\mathcal{O}_k/\mathfrak{a}|^{\dim \mathbf{G}}$$

representatives in $\mathrm{Orbit}_{[\Lambda:\triangle]}(\triangle)$.

Though Theorem 3.3 is stated only for principal arithmetic lattices, it holds for congruence subgroups of a principal arithmetic lattice. When $\mathbf{G}$ is not simply connected, the result holds with a multiplicative error depending only on the cardinality of the kernel of the isogeny $\widetilde{\mathbf{G}} \longrightarrow \mathbf{G}$ between $\mathbf{G}$ and its simply connected representative $\widetilde{\mathbf{G}}$. Likewise, for groups $\mathbf{G}$ which are not connected, we get a multiplicative error depending on the cardinality of $\pi_0(\mathbf{G})$. In total, for any $\mathbf{G}$ with $\Lambda, \triangle$ as above, the number of distinct Λ–conjugacy classes of subgroups of Λ isomorphic to $\triangle$ is bounded above by

$$C'' \, |\mathcal{O}_k/\mathfrak{a}|^{\dim \mathbf{G}}$$

where C'' is a constant depending only on $\mathbf{G}$.

In the case the isogeny class contains $\mathrm{SO}_0(2,1)$, the manifolds associated to a pair of finite index subgroups $\triangle_1, \triangle_2$ of Λ are isometric if and only if $\triangle_1$ resides in $\mathrm{Orbit}_{[\Lambda:\triangle_2]}(\triangle_2)$. Moreover, the proof of Theorem 3.3 extends to this setting. As we are concerned with isometry classes of manifolds and not isomorphism classes of subgroups, the lack of strong rigidity in this setting is not an issue. Consequently, we do not omit this case, though our proofs must be modified slightly, as we often consider isomorphism classes of subgroups instead of isometry classes of covers.

4 Sunada's method and discrete isospectral deformations

For a finite group N, we say subgroups H_1, H_2 of N are *almost conjugate* if for every conjugacy class $[n]$ of N,

$$|H_1 \cap [n]| = |H_2 \cap [n]|.$$

The triple (N, H_1, H_2) is sometimes called a *Sunada triple* (see [13] and loc. cit. for some constructions) and their importance in this article is in a technique developed by Sunada [39] for constructing isospectral Riemannian manifolds.

Theorem 4.1 (Sunada; [39]). *Let M be a closed Riemannian n–manifold with fundamental group $\pi_1(M)$ and (N,H_1,H_2) a Sunada triple for which there exists a surjective homomorphism $\rho\colon \pi_1(M) \longrightarrow N$. If M_1 and M_2 are the corresponding finite covers associated to the finite index subgroups $\rho^{-1}(H_1)$ and $\rho^{-1}(H_2)$ of $\pi_1(M)$, then M_1 and M_2 are isospectral.*

Let $\{N_i\}$ a family of finite groups and $\{H_{i,1},\dots,H_{i,r_i}\}$ a family of pairwise almost conjugate, nonconjugate subgroups of N_i. We say Γ admits a *discrete isospectral deformation* with respect to

$$\left\{\rho_i, N_i, \left\{H_{i,j}\right\}_j\right\}_i$$

if there exists a finite index subgroup Γ_0 of Γ and surjective homomorphisms $\rho_i\colon \Gamma_0 \longrightarrow N_i$. For future reference, we refer to such families

$$\left\{N_i, \left\{H_{i,j}\right\}_j\right\}_i$$

as *isospectral deformation families* and say a family is *unbounded* when $\{r_i\}_i$ is unbounded. For a group Γ and finite index subgroup $\triangle$ of Γ, let $\iota_\ell(\triangle)$ denote the number of index ℓ subgroups of Γ isomorphic to $\triangle$. If Γ admits a discrete isospectral deformation with respect to

$$\left\{\rho_i, N_i, \left\{H_{i,j}\right\}_j\right\}_i,$$

we say the deformation is *nontrivial* if there exists $i_0 > 0$ such that for some j

$$\iota_{[\Gamma:\rho_{i_0}^{-1}(H_{i_0,j})]}(\rho_{i_0}^{-1}(H_{i_0,j})) < r_{i_0}.$$

If in addition the sequence

$$\frac{r_i}{\sup_j \iota_{[\Gamma:\rho_i^{-1}(H_{i,j})]}(\rho_i^{-1}(H_{i,j}))}$$

is unbounded, we say the deformation is *unbounded*. Our interest in isospectral deformation families is revealed in the formal proposition.

Proposition 4.2. **(a)** *If M is a closed Riemannian n–manifold and $\pi_1(M)$ admits a nontrivial discrete isospectral deformation, then M has a pair of finite isospectral, nonisometric covers.*

(b) *If M is a closed Riemannian n–manifold and $\pi_1(M)$ admits an unbounded discrete isospectral deformation, then for every t, M has t isospectral, nonisometric finite covers.*

Proof. Let
$$\left\{\rho_i, N_i, \left\{H_{i,j}\right\}_j\right\}_i$$
be a nontrivial discrete isospectral deformation of $\pi_1(M)$. Without loss of generality, we may assume that ρ_i is defined on $\pi_1(M)$. By Theorem 4.1, the covers corresponding to the subgroups $\rho_i^{-1}(H_{i,j})$ for $1 \leq j \leq r_i$ are isospectral and we denote the associated manifolds by $M_{i,j}$. Note if M_{i,j_1} and M_{i,j_2} are isometric, then $\pi_1(M_{i,j_1})$ and $\pi_1(M_{i,j_2})$ are isomorphic. If $\iota(M_{i,j})$ denotes the number of covers of M isometric to $M_{i,j}$ with the same number of sheets, then

$$\iota(M_{i,j}) \leq \iota_{[\pi_1(M):\rho_i^{-1}(H_{i,j})]}(\rho_i^{-1}(H_{i,j})). \tag{1}$$

By assumption this discrete isospectral deformation is nontrivial and hence there exist a pair of covers $M_{i_0,j}$ and $M_{i_0,j'}$ which are not isometric. Specifically for some i_0 and j
$$\iota_{[\pi_1(M):\rho_{i_0}^{-1}(H_{i_0,j})]}(\rho_{i_0}^{-1}(H_{i_0,j})) < r_{i_0},$$
and thus according to the inequality (1), there must exist a pair of nonisometric manifolds $M_{i_0,j}$ and $M_{i_0,j'}$. For (b), we partition the set $\left\{\pi_1(M_{i,j})\right\}$ into isomorphism classes. Since this isospectral deformation is unbounded, the number of partition sets is unbounded as a function of i. To see this note if the number of partitions were bounded above by s, we would have

$$s\left(\sup_j\left\{\iota_{[\pi_1(M):\rho_i^{-1}(H_{i,j})]}(\rho_i^{-1}(H_{i,j}))\right\}\right) \geq r_i$$

for all i which contradicts the unbounded assumption. For each pair of partition sets we obtain a pair of isospectral nonisometric finite covers. Hence, for any $t > 0$ we can find t pairwise isospectral nonisometric finite covers of M by selecting $i(t)$ such that
$$t\left(\sup_j\left\{\iota_{[\pi_1(M):\rho_{i(t)}^{-1}(H_{i(t),j})]}(\rho_{i(t)}^{-1}(H_{i(t),j}))\right\}\right) < r_{i(t)}.$$
□

As we make use of this below, a few remarks are in order on the above proof. The statement of (b) produces arbitrarily large families of pairwise isospectral, pairwise nonisometric covers of M. Indeed, the proof produces arbitrarily large families of finite, isospectral covers such that the probability any two are isometric is exceedingly small. Specifically, if $\beta_{i,j}$ is the number of manifolds $M_{i,j'}$ which are isometric to $M_{i,j}$, then the probability $M_{i,j}$ is isometric to $M_{i,j'}$ for some randomly selected $M_{i,j'}$ is $\beta_{i,j}/r_i$. The proof shows the existence of an infinite sequence i_ℓ such that for each i_ℓ and all j

$$\lim_{i_\ell} \frac{\beta_{i_\ell,j}}{r_{i_\ell}} = 0.$$

Thus the probability two randomly selected covers $M_{i_\ell,j}$ and $M_{i_\ell,j'}$ are isometric is exceedingly small and in the limit produces isospectral nonisometric covers with probability one. Despite the fact this result generically produces examples, it is nevertheless nonconstructive. This can be compared with the recent paper of Belolipetsky and Lubotzky [2] who gave a probabilistic construction for the existence of closed hyperbolic n–manifolds with a specified symmetry group.

5 Heisenberg groups over finite fields and rings

Much of this material can be found in [7]. For a commutative ring R with identity 1 and $m > 2$, $\mathrm{GL}(m;R)$ has the subgroups

$$\mathfrak{N}_3(R) = \left\{ \begin{pmatrix} 1 & x & t & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_{m-3} \end{pmatrix} \; : \; x,y,t \in R \right\}$$

and

$$H(R) = \left\{ \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_{m-3} \end{pmatrix} \; : \; x \in R \right\}$$

called the *3–dimensional Heisenberg group over R* and the *horizontal subgroup* of $\mathfrak{N}_3(R)$. Spatzier [37] used $\mathfrak{N}_3(\mathbb{F}_3)$ in his construction of isospectral nonisometric locally symmetric manifolds. Together with an abelian group $(\mathbf{Z}/3\mathbf{Z})^3$ in the alternating group Alt(27), $(\mathrm{Alt}(27), \mathfrak{N}_3(\mathbb{F}_3), (\mathbf{Z}/3\mathbf{Z})^3)$ forms a Sunada triple—see also Sunada [39] who used this triple in isospectral constructions. It is well known that Alt(27) is a subgroup of the Weyl groups of the classical groups A_{26}, B_{13}, C_{27}, and D_{13}. Spatzier used this and the Strong Approximation Theorem to get surjections of the fundamental group of any closed locally symmetric manifold (whose complexification contains A_{26}, B_{13}, C_{27}, or D_{13}) onto groups which contain Alt(27). The isometry problem was dealt with by explicitly showing the pullbacks of the groups $\mathfrak{N}_3(\mathbb{F}_3)$ and $(\mathbf{Z}/3\mathbf{Z})^3$ were not isomorphic.

5.1 Almost linear maps and twisted subgroups Any R–module homomorphism $f\colon R^m \longrightarrow R$ will be called an *R–linear map*. More generally, if f is an additive homomorphism and for any free R–basis $\{s_1, \dots, s_m\}$ of R^m the equality

$$f\left(\sum_{\ell=1}^{m} r_\ell s_\ell\right) = \sum_{\ell=1}^{r} \lambda_\ell r_\ell \text{ for some } \lambda_\ell \in R \text{ and all } r_1, \dots, r_m$$

holds, we call f an *almost R–linear map*. We denote the R–module of all almost R–linear maps by $\mathcal{AL}_m(R)$ and the R–submodule of those R–linear maps by $\mathcal{L}_m(R)$. For simplicity, when $m = 1$, the subscript m is suppressed.

For any almost linear map f on R^2, we define the *f–twisted horizontal group* to be

$$^{f}H(R) = \left\{ \begin{pmatrix} 1 & x & f(x,0) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_{m-3} \end{pmatrix} \;:\; x \in R \right\}.$$

This subgroup depends only on $f(x) = f(x,0)$ which is (almost) R–linear when $f(x,y)$ is. In particular, $f(x) \in \mathcal{AL}(R)$ when $f(x,y) \in \mathcal{AL}_2(R)$ and $f(x) \in \mathcal{L}(R)$ when $f(x,y) \in \mathcal{L}_2(R)$. In regard to isospectrality, the importance of these subgroups is revealed in the following proposition.

Proposition 5.1 ([7]). *$^{f}H(R)$ and $^{g}H(R)$ are almost conjugate for all almost R–linear maps f, g, and $^{f}H(R)$, $^{g}H(R)$ are conjugate in $\mathfrak{N}_3(R)$ if and only if $f - g$ is R–linear.*

To make use of these almost conjugate subgroups, we require their existence in great abundance. This is given by the following lemma of [7].

Lemma 5.2. *For $R = \mathbb{F}_q$ and $q = p^n$,*

$$|\mathcal{AL}(R)/\mathcal{L}(R)| = p^{n(n-1)}.$$

Corollary 5.3.

$$\left\{ \mathfrak{N}_3(\mathbb{F}_q), \left\{ {}^{f}H(\mathbb{F}_q) \right\}_f \right\}_p$$

is an unbounded isospectral deformation family where $\mathbb{F}_q$ ranges over fields of order p^n, with n fixed.

5.2 The family of examples Let k be a number field and $\mathcal{P}$ an infinite subset of $\mathcal{P}(k)$. Removing excess ideals in $\mathcal{P}$, we can arrange it such that if two ideals $\mathfrak{p},\mathfrak{q}$ in $\mathcal{P}$ produce residue fields with identical characteristic, then $\mathfrak{p}=\mathfrak{q}$. With this assumption, we order the ideals in $\mathcal{P}$ according to their characteristic. Specifically, $\mathfrak{p}<\mathfrak{q}$ if and only if

$$\mathrm{char}(\mathcal{O}_k/\mathfrak{p}) < \mathrm{char}(\mathcal{O}_k/\mathfrak{q}).$$

With this we order the elements of $\mathcal{P}$ by $\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3,\dots\}$, and call an infinite subset of prime ideals with this ordering an *ordered prime ideal set*.

For an ordered prime ideal set $\mathcal{P}$, we define an ideal set $\mathcal{I}(\mathcal{P})=\{\mathfrak{b}_i\}_i$ inductively by

$$\mathfrak{b}_1=\mathfrak{p}_1, \quad \mathfrak{b}_{i+1}=\mathfrak{b}_i\mathfrak{p}_{i+1},$$

and we call the set of these ideals $\mathcal{I}(\mathcal{P})$ the *ideal set associated to* $\mathcal{P}$.

Lemma 5.4. *Let ideal set* $\mathcal{I}(\mathcal{P})$ *with* $\mathcal{O}_k/\mathfrak{p}\neq\mathbb{F}_p$ *for all* $\mathfrak{p}\in\mathcal{P}$. *Then*

$$\left\{\mathfrak{N}_3(\mathcal{O}_k/\mathfrak{b}_i),\left\{\,{}^{f}H(\mathcal{O}_k/\mathfrak{b}_i)\right\}_f\right\}_{\mathfrak{b}_i\in\mathcal{I}(\mathcal{P})}$$

is an unbounded isospectral deformation family.

Proof. By the Chinese remainder theorem

$$\mathcal{O}_k/\mathfrak{b}_i=\prod_{\ell=1}^{i}\mathcal{O}_k/\mathfrak{p}_i$$

and

$$\mathfrak{N}_3(\mathcal{O}_k/\mathfrak{b}_i)=\prod_{\ell=1}^{i}\mathfrak{N}_3(\mathcal{O}_k/\mathfrak{p}_\ell).$$

Given an $(\mathcal{O}_k/\mathfrak{b}_i)$–almost linear map f, by additivity

$$f(m(1,1,\dots,1))=mf(1,1,\dots,1).$$

Since $\mathrm{char}(\mathcal{O}_k/\mathfrak{p}_\ell)$ is distinct for each prime $\mathfrak{p}_\ell$, if f is nonzero, f restricted to the factor $\mathcal{O}_k/\mathfrak{p}_\ell$ must take values in $\mathcal{O}_k/\mathfrak{p}_\ell$. In particular, f splits as a direct sum

$$f=\prod_{\ell=1}^{i}f_\ell$$

where each $f_\ell\in\mathcal{AL}(\mathcal{O}_k/\mathfrak{p}_\ell)$. Thus,

$$\mathcal{AL}(\mathcal{O}_k/\mathfrak{b}_i)=\prod_{\ell=1}^{i}\mathcal{AL}(\mathcal{O}_k/\mathfrak{p}_\ell).$$

The linear case

$$\mathcal{L}(\mathcal{O}_k/\mathfrak{b}_i) = \prod_{\ell=1}^{i} \mathcal{L}(\mathcal{O}_k/\mathfrak{p}_\ell)$$

is identical. According to Lemma 5.2

$$|\mathcal{AL}(\mathcal{O}_k/\mathfrak{p}_\ell)/\mathcal{L}(\mathcal{O}_k/\mathfrak{p}_\ell)| = p_\ell^{n_\ell(n_\ell-1)}$$

where $|\mathcal{O}_k/\mathfrak{p}_\ell| = p_\ell^{n_\ell}$, and so

$$|\mathcal{AL}(\mathcal{O}_k/\mathfrak{b}_i)/\mathcal{L}(\mathcal{O}_k/\mathfrak{b}_i)| = \prod_{\ell=1}^{i} p_\ell^{n_\ell(n_\ell-1)}. \tag{2}$$

Thus

$$\left\{\mathfrak{N}_3(\mathcal{O}_k/\mathfrak{b}_i), \left\{ {}^fH(\mathcal{O}_k/\mathfrak{b}_i)\right\}_f\right\}_{\mathfrak{b}_i\in\mathcal{I}(\mathcal{P})}$$

is an unbounded isospectral deformation family since

$$n_\ell(n_\ell-1) > 1. \tag{3}$$

Specifically,

$$\begin{aligned}|\mathcal{AL}(\mathcal{O}_k/\mathfrak{b}_i)/\mathcal{L}(\mathcal{O}_k/\mathfrak{b}_i)| &= \prod_{\ell=1}^{i} p_\ell^{n_\ell(n_\ell-1)} \geq \prod_{\ell=1}^{i} 2^{n_\ell(n_\ell-1)} \\ &= 2^{i(n_\ell(n_\ell-1))} \geq 2^i.\end{aligned}$$

□

5.3 Heisenberg deformations For f,g in $\mathcal{AL}(R)$ with non-linear $f-g$ it could be that ${}^fH(R)$ and ${}^gH(R)$ are conjugate in $\mathrm{GL}(m;R)$; this should almost never happen. We denote the $\mathfrak{N}_3(R)$–conjugacy classes of almost conjugate twisted horizontal subgroups by $\mathcal{AC}(R)$ which by Proposition 5.1 are in bijection with the elements of $\mathcal{AL}(R)/\mathcal{L}(R)$. The $\mathrm{GL}(m;R)$–conjugacy class of ${}^fH(R)$ viewed as a subset of $\mathcal{AC}(R)$ is defined to be

$$\mathrm{Orbit}_m({}^fH(R)) = \left\{ {}^gH(R) \in \mathcal{AC}(R) \;:\; \gamma^{-1}\, {}^fH(R)\gamma = {}^gH(R)\right\}.$$

The equivalence classes $\mathcal{AC}(R)$ under the conjugate action of $\mathrm{GL}(m;R)$ will be denoted by ${}^m\mathcal{AC}(R)$.

Proposition 5.5. *Let $\mathcal{I}(\mathcal{P})$ be an ideal set associated to an ordered prime ideal set $\mathcal{P}$.*

(a)

$$|\,{}^m\mathcal{AC}(\mathcal{O}_k/\mathfrak{b}_i)| \geq \prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(m^2+1)}.$$

(b) *There exists a constant $C > 0$ such that if $n_\ell \geq C$, then*

$$\{\mathrm{GL}(m;\mathcal{O}_k/\mathfrak{b}_i),\ {}^m\mathcal{AC}(\mathcal{O}_k/\mathfrak{b}_i)\}_{\mathfrak{b}_i \in \mathcal{I}(\mathcal{P})}$$

is an unbounded isospectral deformation family.

Proof. By (2), there are at least

$$\prod_{\ell=1}^{i} p_\ell^{n_\ell(n_\ell-1)}$$

distinct twisted horizontal subgroups of $\mathfrak{N}_3(\mathcal{O}_k/\mathfrak{b}_i)$. Therefore,

$$\begin{aligned}
|\,{}^m\mathcal{AC}(\mathcal{O}_k/\mathfrak{b}_i)| &\geq \frac{|\mathcal{AC}(\mathcal{O}_k/\mathfrak{b}_i)|}{\max_f |\mathrm{Orbit}_m(\,{}^fH(\mathcal{O}_k/\mathfrak{b}_i))|} \\
&\geq \frac{|\mathcal{AC}(\mathcal{O}_k/\mathfrak{b}_i)|}{\max_f [\mathrm{GL}(m;\mathcal{O}_k/\mathfrak{b}_i) :\ {}^fH(\mathcal{O}_k/\mathfrak{b}_i)]} \\
&\geq \frac{\prod_{\ell=1}^{i} p_\ell^{n_\ell(n_\ell-1)}}{\max_f [\mathrm{GL}(m;\mathcal{O}_k/\mathfrak{b}_i) :\ {}^fH(\mathcal{O}_k/\mathfrak{b}_i)]} \\
&\geq \frac{\prod_{\ell=1}^{i} p_\ell^{n_\ell(n_\ell-1)}}{|\mathrm{GL}(m;\mathcal{O}_k/\mathfrak{b}_i)|} \geq \frac{\prod_{\ell=1}^{i} p_\ell^{n_\ell(n_\ell-1)}}{|(\mathcal{O}_k/\mathfrak{b}_i)|^{m^2}} \\
&= \frac{\prod_{\ell=1}^{i} p_\ell^{n_\ell(n_\ell-1)}}{\prod_{\ell=1}^{i} p_\ell^{n_\ell m^2}} = \prod_{\ell=1}^{i} p_\ell^{n_\ell(n_\ell-1)-n_\ell m^2} \\
&= \prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+m^2)}.
\end{aligned}$$

There exists a smallest integer C such that

$$C^2 - C(m^2+1) \geq 1, \tag{4}$$

and for all $n_\ell \geq C$

$$n_\ell^2 - n_\ell(m^2+1) \geq 1.$$

Arguing as in the derivation of Lemma 5.4, we conclude there are at least 2^i distinct $\mathrm{GL}(m;\mathcal{O}_k/\mathfrak{b}_i)$–conjugacy classes of twisted horizontal subgroups of $\mathfrak{N}_3(\mathcal{O}_k/\mathfrak{b}_i)$. Hence,

$$\{\mathrm{GL}(m;\mathcal{O}_k/\mathfrak{b}_i),\ {}^m\mathcal{AC}(\mathcal{O}_k/\mathfrak{b}_i)\}_{\mathfrak{b}_i \in \mathcal{I}(\mathcal{P})}$$

is an unbounded isospectral deformation family. □

For a k–algebraic group $\mathbf{G}$ such that $\mathbf{G}(\mathcal{O}_k/\mathfrak{b}_i)$ contains $\mathfrak{N}_3(\mathcal{O}_k/\mathfrak{b}_i)$, Proposition 5.5 holds with

$$\left|\,{}^{\mathbf{G}}\mathcal{AC}(\mathcal{O}_k/\mathfrak{b}_i)\right| \geq \prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+\dim \mathbf{G})}.$$

Here ${}^{\mathbf{G}}\mathcal{AC}(\mathcal{O}_k/\mathfrak{b}_i)$ denotes the equivalence classes of $\mathcal{AC}(\mathcal{O}_k/\mathfrak{b}_i)$ under $\mathbf{G}(\mathcal{O}_k/\mathfrak{b}_i)$–conjugation.

6 Isospectral deformations for large groups

Recall a group Γ is *large* if there exists a finite index subgroup Γ_0, a nonabelian free group F_r of rank r, and a surjective homomorphism $\rho\colon \Gamma_0 \longrightarrow F_r$. We require the following which is a generalization of [7] for surface groups.

Lemma 6.1. *Every large group admits a discrete isospectral deformation.*

Proof. Consider the isospectral deformation family

$$\left\{\mathfrak{N}_3(\mathbb{F}_q), \left\{\,{}^{f}H(\mathbb{F}_q)\right\}_f\right\}_p,$$

where $q = p^n$ and let the family vary over all primes p. As the group $\mathfrak{N}_3(\mathbb{F}_q)$ can be generated by $2n$ elements and Γ is large, there exists a finite index subgroup Γ_0 of Γ and sequence

$$\Gamma_0 \overset{\rho}{\twoheadrightarrow} F_{2n} \overset{\pi_p}{\twoheadrightarrow} \mathfrak{N}_3(\mathbb{F}_q)\ .$$

By Corollary 5.3,

$$\left\{\mathfrak{N}_3(\mathbb{F}_q), \left\{\,{}^{f}H(\mathbb{F}_q)\right\}_f\right\}_p$$

is an isospectral deformation family, and so Γ admits a discrete isospectral deformation with respect to

$$\left\{\pi_p \circ \rho, \mathfrak{N}_3(\mathbb{F}_q), \left\{\,{}^{f}H(\mathbb{F}_q)\right\}_f\right\}_p.$$

□

Proposition 6.2 (Unboundedness)**.** *If Λ is an irreducible, large, nonarithmetic lattice in a semisimple Lie group G with finite center and no compact factors, then Λ admits an unbounded discrete isospectral deformation.*

Proof. By Lemma 6.1, Λ admits a discrete isospectral deformation with

$$\left\{\pi_p \circ \rho, \mathfrak{N}_3(\mathbb{F}_q), \left\{ {}^f H(\mathbb{F}_q)\right\}_f\right\}_p.$$

In this family, n is fixed and p ranges over the set of all prime integers. By Lemma 5.2 we have $p^{n(n-1)}$ almost conjugate nonconjugate subgroups of $\mathfrak{N}_3(\mathbb{F}_q)$ given by the twisted horizontal groups ${}^f H(\mathbb{F}_q)$. Setting

$${}^f\triangle_p = r_p^{-1}({}^f H(\mathbb{F}_q)),$$

by construction we have $p^{n(n-1)}$ pullbacks of twisted horizontal subgroups which are not conjugate in Λ_0. By Lemma 3.2, the number of non-isomorphic subgroups of distinct pullbacks of twisted horizontal subgroups in Λ is at least

$$\frac{p^{n(n-1)}}{\sup_f \iota_{[\Lambda_0:\, {}^f\triangle_p]}({}^f\triangle_p)} \geq \frac{p^{n(n-1)}}{C'[\mathrm{Comm}_G(\Lambda):\Lambda_0]}$$

for a constant C'. As both $[\mathrm{Comm}_G(\Lambda):\Lambda_0]$, n are constant and p can be arbitrarily large, this number is unbounded, as required. □

Despite the simplicity of Proposition 6.2, this produces the first examples of isospectral, nonisometric complex hyperbolic 2–manifolds. In high dimensions, this produces the first examples of arbitrarily large collections of isospectral nonisometric real hyperbolic n–manifolds. The former is achieved with a lattice of Livné's thesis [15] while the latter are obtained from the lattices of Gromov–Piatetski-Shapiro [12].

7 Discrete isospectral deformation for arithmetic lattices

The goal of this section is the construction of unbounded discrete isospectral deformations for principle arithmetic lattices in semisimple Lie groups.

7.1 Heisenberg groups in classical groups The three dimensional Heisenberg group $\mathfrak{N}_3$ is the simply connected connected nilpotent Lie group of real dimension three given by

$$\mathfrak{N}_3 = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;:\; x,y,t \in \mathbf{R} \right\}.$$

Proposition 7.1. *If* $\mathbf{G}$ *a real noncompact classical group, then there exists a continuous injection* $\psi\colon \mathfrak{N}_3(\mathbb{A}_F^f) \longrightarrow \mathbf{G}(\mathbb{A}_F^f)$.

Proof. For every prime p, $\mathrm{SO}(2,1;\mathbf{Z}/p^j\mathbf{Z})$ contains a unipotent subgroup which is $\mathrm{GL}(3;\mathbf{Z}/p^j\mathbf{Z})$–conjugate to $\mathfrak{N}_3(\mathbf{Z}/p^j\mathbf{Z})$ (see [1, Ch. 7]). Consequently, we have a continuous injection $\psi_p\colon \mathfrak{N}_3(\mathbf{Q}_p) \longrightarrow \mathrm{SO}(2,1;\mathbf{Q}_p)$. Indeed, this holds for any group $\mathbf{H}$ isogenous to $\mathrm{SO}(2,1)$ since $\mathfrak{N}_3$ is unipotent. To obtain the injection for any classical group $\mathbf{G}$, we simply note that $\mathbf{H}(\mathbf{Q}_p)$ injects into $\mathbf{G}(\mathbf{Q}_p)$ for some $\mathbf{H}$ isogenous to $\mathrm{SO}(2,1)$. In total, this yields the injection $\mathfrak{N}_3(\mathbb{A}_F^f)$ into $\mathbf{G}(\mathbb{A}_F^f)$ for any finite extension $F/\mathbf{Q}$. □

Since $\mathfrak{N}_3$ is unipotent, this also holds for $\mathbf{G}'$ that is isogenous (over F) to a real noncompact classical group. For an F–form $\mathbf{H}$ of $\mathbf{G}$, a more delicate argument is required since the isomorphism between $\mathbf{G}$ and $\mathbf{H}$ is defined over a finite extension L/F.

Proposition 7.2. *Let* $\mathbf{H}$ *be an* F*–form of a real noncompact classical group* $\mathbf{G}$*. Then there exists a finite number of valuations* S *of* F *and a continuous injection* $\psi\colon \mathfrak{N}_3(\mathbb{A}_F(S)) \longrightarrow \mathbf{H}(\mathbb{A}_F(S))$.

Proof. To begin, there is a bijection between the F–isomorphism classes of F–forms of $\mathbf{G}$ and the Galois cohomology set

$$H^1(\mathrm{Gal}(\overline{F}/F);\mathrm{Aut}_F(\mathbf{G}))$$

where $\overline{F}$ is the algebraic closure of F (see [31]). This set is a direct limit

$$\lim_{L/F} H^1(\mathrm{Gal}(L/F);\mathrm{Aut}_F(\mathbf{G}))$$

over the finite Galois extensions L/F. For each valuation $\nu \in V(F)$, each class in $H^1(\mathrm{Gal}(L/F);\mathrm{Aut}_F(\mathbf{G}))$ yields a class in $H^1(\mathrm{Gal}(L_{\nu_L}/F_\nu);\mathrm{Aut}_{F_\nu}(\mathbf{G}))$ where ν_L is an extension of ν to L. For all but finitely many $\nu \in V(F)$, this class corresponds to the trivial class associated to $\mathbf{G}$ and thus produces an injection

$$H^1(\mathrm{Gal}(\overline{F}/F);\mathrm{Aut}_F(\mathbf{G})) \subset \bigoplus_{\nu\in V(F)} H^1(\mathrm{Gal}(\overline{F_\nu}/F_\nu);\mathrm{Aut}_{F_\nu}(\mathbf{G}))).$$

In particular, $\mathbf{G}(F_\nu)$ and $\mathbf{H}(F_\nu)$ are F_ν–isomorphic for all but finitely many $\nu \in V(F)$—in many cases the Hasse Norm Theorem or the Albert–Hasse–Brauer–Noether Theorem for Brauer groups [29] imply this. Consequently, there exists a finite set of valuation S of $V(F)$ such that $\mathbf{H}(\mathbb{A}_F(S))$ and $\mathbf{G}(\mathbb{A}_F(S))$ are isomorphic and so by Proposition 7.1, $\mathbf{H}(\mathbb{A}_F(S))$ contains a copy of $\mathfrak{N}_3(\mathbb{A}_F(S))$. □

Again, since $\mathfrak{N}_3$ is unipotent, if $\mathbf{G}'$ is isogenous to a real, noncompact classical group and $\mathbf{H}'$ is an F–form of $\mathbf{G}'$, there exists a finite set of valuations S of F and an injection of $\mathfrak{N}_3(\mathbb{A}_F(S))$ into $\mathbf{H}'(\mathbb{A}_F(S))$.

7.2 Unboundedness In our constructions, we will require the field of definition F of an F–form $\mathbf{H}$ of a classical group $\mathbf{G}$ to satisfy one of the following inequalities:

$$[F:\mathbf{Q}]^2 - [F:\mathbf{Q}](1+2\dim \mathbf{G}) \geq 1 \tag{5}$$

$$[F:\mathbf{Q}]^2 - [F:\mathbf{Q}](1+\dim \mathbf{G}) \geq 1. \tag{6}$$

The following result produces unbounded discrete isospectral deformations for a wide class of lattices.

Theorem 7.3 (Unboundedness)**.** *Let* $\mathbf{G}$ *be a real, simply connected, noncompact classical group.*

(a) *Let* Λ *be a principal arithmetic lattice in* $\mathbf{G}$ *defined over* F *with* F*–form* $\mathbf{H}$ *such that* F *satisfies (5). Then*

$$\left\{ r_{\mathfrak{b}_i}, \mathbf{H}(\mathcal{O}_F/\mathfrak{b}_i),\ {}^{\mathbf{H}}\mathcal{AC}(\mathcal{O}_F/\mathfrak{b}_i) \right\}_{\mathcal{I}(\mathcal{P}(\Lambda))}$$

is unbounded for some prime ideal set $\mathcal{P}(\Lambda)$.

(b) *Let* Λ *be a cocompact nonarithmetic lattice in* $\mathbf{G}$ *with trace field* F*, trace ring* R*, and with* F *satisfying (6). Then*

$$\left\{ r_{\mathfrak{b}_i}, \mathbf{G}(R/\mathfrak{b}_i),\ {}^{\mathbf{G}}\mathcal{AC}(R/\mathfrak{b}_i) \right\}_{\mathcal{I}(\mathcal{P}(\Lambda))}$$

is unbounded for some ideal set $\mathcal{P}(\Lambda)$.

Proof. By Proposition 7.2, there exists a finite set of valuations S of F such that $\mathfrak{N}_3(\mathbb{A}_F(S)) < \mathbf{H}(\mathbb{A}_F(S))$. By the Strong Approximation Theorem, the closure of Λ in $\widetilde{\mathbf{H}}(\mathbb{A}_F(S))$ under the diagonal embedding is compact and open. Therefore, the closure of $\Lambda \cap \mathfrak{N}_3(\mathbb{A}_F(S))$ in an open, compact subgroup of $\mathfrak{N}_3(\mathbb{A}_F(S))$. Hence, there exists a finite set of valuations T such that the closure of Λ in $\mathbf{G}(\mathbb{A}_F(T))$ contains $\mathfrak{N}_3(\mathbb{A}_F(T))$. Thus, there exists a finite set of prime ideals $\mathcal{P}_{bad}$ of $\mathcal{P}(R)$ such that if $\mathfrak{p}$ is not in $\mathcal{P}_{bad}$, then $r_{\mathfrak{p}^j}(\Lambda)$ contains a unipotent subgroup isomorphic to $\mathfrak{N}_3(\mathcal{O}_F/\mathfrak{p}^j)$.

To produce nontrivial (and further unbounded) deformations, some conditions are required on the prime ideals in $\mathcal{P}'(\Lambda) = \mathcal{P}(F) \setminus \mathcal{P}_{bad}$. By the Cebotarev Density Theorem, there exists infinitely many prime ideals $\mathfrak{p}$ of $\mathcal{P}(F)$ such that

$$[\mathcal{O}_F/\mathfrak{p} : \mathbb{F}_p] = [F:\mathbf{Q}].$$

From this, let $\mathcal{P}''(\Lambda)$ comprise the set of all such prime ideals.

Take $\mathcal{P}(\Lambda)$ to be the intersection $\mathcal{P}'(\Lambda)$ and $\mathcal{P}''(\Lambda)$ with excess prime ideals removed. Note, as the former is cofinite and the latter is infinite, this set is infinite. Now, consider the discrete isospectral deformation

$$\left\{ r_{\mathfrak{b}_i}, \mathbf{H}(\mathcal{O}_F/\mathfrak{b}_i),\ {}^{\mathbf{H}}\mathcal{AC}(\mathcal{O}_F/\mathfrak{b}_i) \right\}_{\mathcal{I}(\mathcal{P}(\Lambda))}$$

of Λ, and denote the pullback of the groups ${}^fH(\mathcal{O}_F/\mathfrak{b}_i)$ under $r_{\mathfrak{b}_i}$ by ${}^f\triangle_{\mathfrak{b}_i}$. By the remarks following the proof of Proposition 5.5, there are at least

$$\prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+\dim \mathbf{H})}$$

of these subgroups of Λ that are not conjugate in Λ. By Theorem 3.3, the number of distinct Λ–conjugacy classes of pullbacks of twisted horizontal subgroups in $\mathrm{Orbit}_{[\Lambda:\triangle]}(\triangle)$ is no more than

$$C'' \prod_{\ell=1}^{i} p_\ell^{n_\ell \dim \mathbf{H}}$$

for a constant C'' which depends only on $\mathbf{H}$. Therefore, the number of distinct (i.e., non-isomorphic as abstract groups) pullbacks of twisted horizontal subgroups is at least

$$\begin{aligned}\frac{\prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+\dim \mathbf{H})}}{\sup_f \iota_{[\Lambda:\, {}^f\triangle_{\mathfrak{b}_i}]}({}^f\triangle_{\mathfrak{b}_i})} &\geq \frac{\prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+\dim \mathbf{H})}}{C'' \prod_{\ell=1}^{i} p_\ell^{n_\ell \dim \mathbf{H}}} \\ &= (C'')^{-1} \prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+2\dim \mathbf{H})}.\end{aligned}$$

By our selection of the ideals $\mathfrak{p}_\ell$ and the fact that F satisfies (5), for every ℓ

$$n_\ell^2 - n_\ell(1+2\dim \mathbf{H}) > 1. \tag{7}$$

Therefore, the number of distinct isomorphism classes of pullbacks is at least

$$\begin{aligned}\frac{\prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+\dim \mathbf{H})}}{\sup_f \iota_{[\Lambda:\, {}^f\triangle_{\mathfrak{b}_i}]}({}^f\triangle_{\mathfrak{b}_i})} &\geq (C'')^{-1} \prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+2\dim \mathbf{H})} \\ &\geq (C'')^{-1} \prod_{\ell=1}^{i} 2^{n_\ell^2 - n_\ell(1+2\dim \mathbf{H})} \\ &= (C'')^{-1} 2^{i(n_\ell^2 - n_\ell(1+2\dim \mathbf{H}))} \\ &\geq (C'')^{-1} 2^i.\end{aligned}$$

For part (b), by Weil's Local Rigidity Theorem (see [34, Ch. 6]), we can conjugate Λ into $\mathbf{G}(F)$. As Λ is the fundamental group of a compact manifold, Λ is finitely presentable. Therefore, the ring generated by the traces of Λ is a finite extension $R/\mathcal{O}_F$. By the Strong Approximation Theorem, the closure of Λ in $\mathbf{G}(\mathbb{A}_F^f)$ under the diagonal embedding is compact and open. Therefore, the closure of $\Lambda \cap \mathfrak{N}_3(\mathbb{A}_F^f)$ in an open, compact subgroup of $\mathfrak{N}_3(\mathbb{A}_F^f)$. Hence, there exists a finite set of valuations T such that the closure of Λ in $\mathbf{G}(\mathbb{A}_F(T))$ contains $\mathfrak{N}_3(\mathbb{A}_F(T))$. Thus, there exists a finite set of prime ideals $\mathcal{P}_{bad}$ of $\mathcal{P}(R)$ such that if $\mathfrak{p}$ is not in $\mathcal{P}_{bad}$, then $r_{\mathfrak{p}^j}(\Lambda)$ contains a unipotent subgroup isomorphic to $\mathfrak{N}_3(\mathcal{O}_F/\mathfrak{p}^j)$. We conclude Λ admits a discrete isospectral deformation with

$$\left\{ r_{\mathfrak{b}_i}, \mathbf{G}(R/\mathfrak{b}_i),\ {}^{\mathbf{G}}\mathcal{AC}(R/\mathfrak{b}_i) \right\}_{\mathcal{I}(\mathcal{P}(\Lambda))},$$

where $\mathcal{P}'(\Lambda) = \mathcal{P}(R) \setminus \mathcal{P}_{bad}$. Since F satisfies (6), we have

$$[F:\mathbf{Q}]^2 - [F:\mathbf{Q}](1+\dim \mathbf{G}) \geq 1.$$

By the Cebotarev Density Theorem, there exists an infinite set of ideals $\mathcal{P}'$ in $\mathcal{P}(F)$ such that $[\mathcal{O}_F/\mathfrak{p} : \mathbb{F}_p] = [F:\mathbf{Q}]$. As $\mathcal{P}$ is cofinite, the intersection of $\mathcal{P}$ with $\mathcal{P}'$ is infinite, and removing excess ideals, we obtain an infinite set of ideals $\mathcal{P}(\Lambda)$ and a discrete isospectral deformation for Λ with

$$\left\{ r_{\mathfrak{b}_i}, \mathbf{G}(R/\mathfrak{b}_i),\ {}^{\mathbf{G}}\mathcal{AC}(R/\mathfrak{b}_i) \right\}_{\mathcal{I}(\mathcal{P}(\Lambda))}.$$

We denote the pullback of the groups ${}^fH(R/\mathfrak{b}_i)$ under $r_{\mathfrak{b}_i}$ by ${}^f\triangle_{\mathfrak{b}_i}$. By Proposition 5.5, there are at least

$$\prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+\dim \mathbf{G})}$$

of these subgroups which are not conjugate in Λ where $p_\ell^{n_\ell} = |R/\mathfrak{p}_\ell|$. By Lemma 3.2, the number of distinct pullbacks of twisted horizontal subgroups is at least

$$\begin{aligned}
\frac{\prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+\dim \mathbf{G})}}{\sup_f \iota_{[\Lambda:\, {}^f\triangle_{\mathfrak{b}_i}]}({}^f\triangle_{\mathfrak{b}_i})} &\geq \frac{\prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+\dim \mathbf{G})}}{C'[\mathrm{Comm}_{\mathbf{G}}(\Lambda):\Lambda]} \\
&\geq \frac{\prod_{\ell=1}^{i} 2^{n_\ell^2 - n_\ell(1+\dim \mathbf{G})}}{C'[\mathrm{Comm}_{\mathbf{G}}(\Lambda):\Lambda]}. \\
&\geq \frac{2^{in_\ell^2 - in_\ell(1+\dim \mathbf{G})}}{C'[\mathrm{Comm}_{\mathbf{G}}(\Lambda):\Lambda]}
\end{aligned}$$

By our selection of the ideals in $\mathcal{P}(\Lambda)$ and the weak $\mathbf{G}$–largeness of F,

$$i(n_\ell^2 - n_\ell(1+\dim\mathbf{G})) > i. \tag{8}$$

As $[\mathrm{Comm}_{\mathbf{G}}(\Lambda):\Lambda]$ and C' are constant,

$$\frac{\prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - n_\ell(1+\dim\mathbf{G})}}{\sup_f \iota_{[\Lambda:\, {}^f\triangle_{\mathfrak{b}_i}]}({}^f\triangle_{\mathfrak{b}_i})} \geq \frac{2^{i(n_\ell^2 - n_\ell(1+\dim\mathbf{G}))}}{C'[\mathrm{Comm}_{\mathbf{G}}(\Lambda):\Lambda]} \geq \frac{2^i}{C'[\mathrm{Comm}_{\mathbf{G}}(\Lambda):\Lambda]}$$

is unbounded as needed. □

As the classical groups are not necessarily simply connected nor connected, strong approximation is well known to fail (see [31, Ch. 7.4]). However, since the kernel of the isogeny $\widetilde{\mathbf{G}} \longrightarrow \mathbf{G}_0$ is finite, for every lattice Λ of $\mathbf{G}$, there exists a finite index subgroup Λ_0 of Λ such that the preimage of Λ_0 under the isogeny $\widetilde{\mathbf{G}} \longrightarrow \mathbf{G}$ maps isomorphically to Λ_0. Therefore, we obtain the conclusions of Theorem 7.3 for the group Λ_0 and so for Λ. In addition, this argument is valid for any congruence subgroup $\triangle$ of $\mathbf{H}(\mathcal{O}_F)$ and does not require $\mathbf{H}$ be admissible. Hence, this produces unbounded isospectral deformations for the irreducible lattices in the noncompact factor of the real points of $\mathrm{Res}_{F/\mathbf{Q}}(\mathbf{H})$.

7.3 Exceptional groups For a real, simple, noncompact, exceptional group $\mathbf{G}$, there exists a continuous injections of $\mathfrak{N}_3(\mathbb{A}_{\mathbf{Q}}^f)$ into $\mathbf{G}(\mathbb{A}_{\mathbf{Q}}^f)$. For example, the two real noncompact groups of type F_4 (the split and quasi-split groups) are isomorphic over $\mathbf{Q}_p$ and the unipotent radical of the Borel subgroup of the quasi-split F_4 contains a $\mathbf{Q}$–defined Heisenberg group via a $\mathbf{Q}$–injection of $\mathrm{PSp}(2,1)$. For the other noncompact, simple, exceptional groups, one can check that the p–adic groups always have a $\mathbf{Q}_p$–defined subgroup $\mathbf{Q}_p$–isomorphic to $\mathbf{H}(\mathbf{Q}_p)$ where $\mathbf{H}$ and $\mathrm{SO}(2,1)$ are isogenous. Applying the Strong Approximation Theorem and Theorem 3.3, we construct unbounded discrete isospectral deformations for principal arithmetic lattices as above provided F satisfies (5). Altogether, this yields:

Theorem 7.4. *Let $\mathbf{H}$ be a noncompact simple F–algebraic group such that F satisfies (5). Then for any congruence subgroup $\triangle$ of $\mathbf{H}(\mathcal{O}_F)$, there exists an ordered prime ideal set $\mathcal{P}(\triangle)$ such that*

$$\left\{ r_{\mathfrak{b}_i}, \mathbf{H}(\mathcal{O}_F/\mathfrak{b}_i), \ {}^{\mathbf{H}}\mathcal{AC}(\mathcal{O}_F/\mathfrak{b}_i) \right\}_{\mathcal{I}(\mathcal{P}(\triangle))}$$

is an unbounded discrete isospectral deformation for $\triangle$.

7.4 Semisimple groups The reduction of the semisimple case to the simple case is standard and we briefly explain this here. Let $\mathbf{G}$ be a real k–algebraic semisimple Lie group with finite center and no compact factors. Then $\mathbf{G}$ admits an irreducible lattice if and only if there exists a real absolutely simple L–algebraic group $\mathbf{H}$ such that $\mathrm{Res}_{L/k}(\mathbf{H}) = \mathbf{G}$; in this case $\mathbf{G}$ is called *isotypic*. Consequently, the irreducible principle arithmetic lattice $\mathbf{G}(\mathcal{O}_k)$ of $\mathbf{G}$ admits unbounded discrete isospectral deformations. Indeed, given our remarks following Theorem 7.3, the proof of Theorem 7.3 (a) produces unbounded discrete isospectral deformations for the groups $\mathbf{H}(\mathcal{O}_L)$. By Margulis' Arithmeticity Theorem, these lattices comprise all the commensurability classes for the irreducible lattices of $\mathbf{G}$, as long as $\mathbf{G}$ is not simple and of real rank one. However, this case was dealt with by Theorem 7.3.

For an isotypic, semisimple Lie group G with finite center and no compact factors, there exists a constant r_0 depending only on the complexification of any simple factor of G such that if G has $r \geq r_0$ simple factors, every principal arithmetic lattice of G admits an unbounded discrete isospectral deformation; the constant r_0 is computed using (5). For instance, if

$$G = \prod_{j=1}^{r} \mathrm{PSL}(2;\mathbf{R}),$$

then when $r \geq 8$ every principal arithmetic lattice in G possesses a discrete isospectral deformation. The principal arithmetic compact manifolds produced are quaternionic Shimura modular varieties and are moduli spaces for principally polarized abelian varieties with quaternionic multiplication.

8 Isospectral examples

8.1 Isospectral covers of algebraically large manifolds In order to produce isospectral nonisometric Riemannian manifolds modelled on $\mathbf{G}/\mathbf{K}$, we must ensure the pullbacks of the subgroups in the unbounded discrete isospectral deformation for our principal arithmetic lattice Λ are torsion free. We achieve this with Selberg's lemma (see [5, Prop. 2.2]). Specifically, the following allows us to ensure certain subgroups are torsion free and can be found in [23].

Proposition 8.1. *Let $k/\mathbf{Q}$ be a finite extension and Λ a finitely generated subgroup of $\mathrm{GL}(n;k)$ with unipotent subgroup Γ. Then for all but finitely many prime ideal $r_{\mathfrak{p}}^{-1}(r_{\mathfrak{p}}(\Gamma))$ is torsion free.*

We now state our main result.

Theorem 8.2. *Let M be a closed, irreducible, locally symmetric manifold modelled on a semisimple Lie group G with finite center and no compact factors. If the trace field of $\pi_1(M)$ satisfies (5), then M is commensurable with a manifold M' such that for every $k > 0$, there exists k nonisometric, isospectral finite covers of M'.*

The existence of admissible F–forms of G with $[F : \mathbf{Q}]$ arbitrarily large is well known (see [40]). Consequently, Theorem 1.1 follows at once from Theorem 8.2 and the reduction to the irreducible case.

Proof of Theorem 8.2. Proposition 8.1 implies that off a finite set of primes $p_1, \dots, p_s$, pullbacks of unipotent subgroups under prime power reduction are torsion free. To prove the isospectral deformations constructed in the previous two sections produce torsion free subgroups, we argue as follows. For an F–form $\mathbf{H}$, principal arithmetic lattice $\Lambda = \mathbf{H}(\mathcal{O}_F)$, and F satisfying (5), by Theorem 7.3, Λ admits an unbounded discrete isospectral deformation with the deformation family

$$\left\{r_{\mathfrak{b}_i}, \mathbf{H}(\mathcal{O}_F/\mathfrak{b}_i),\ {}^{\mathbf{H}}\mathcal{AC}(\mathcal{O}_F/\mathfrak{b}_i)\right\}_{\mathcal{I}(\mathcal{P}(\Lambda))}$$

The subgroups ${}^{f}H(\mathcal{O}_F/\mathfrak{b}_i)$ are unipotent and thus by Proposition 8.1, for all but finitely many prime ideals in $\mathcal{P}(\Lambda)$, $r_{\mathfrak{p}^j}^{-1}({}^{f}H(\mathcal{O}_F/\mathfrak{p}))$ is torsion free for all j and f.

Let $\pi_1(M)$ be nonarithmetic with trace ring R and trace field F in $\mathbf{G}$ and assume F satisfies (6). By Theorem 7.3 (b), $\pi_1(M)$ admits an unbounded discrete isospectral deformation with

$$\left\{r_{\mathfrak{b}_i}, \mathbf{G}(R/\mathfrak{b}_i),\ {}^{\mathbf{G}}\mathcal{AC}(R/\mathfrak{b}_i)\right\}_{\mathcal{I}(\mathcal{P}(\Lambda))}$$

where $\pi_1(M)$ is contained in $\mathbf{G}(R)$ for some finite extension ring $R/\mathcal{O}_F$. By Proposition 4.2 (b), we obtain the desired isospectral covers.

Otherwise, we may assume M is arithmetic with field of definition satisfying (5). In this case, there exists an absolutely simple F–algebraic group $\mathbf{H}$ such that $\pi_1(M)$ is commensurable with $\mathbf{H}(\mathcal{O}_F)$. Viewing $\pi_1(M)$ as a subgroup of $\mathbf{H}$, the subgroup $\pi_1(M) \cap \mathbf{H}(\mathcal{O}_F)$ is finite index in $\mathbf{H}(\mathcal{O}_F)$. To apply Theorem 7.3 to $\pi_1(M) \cap \mathbf{H}(\mathcal{O}_F)$, we require this subgroup be congruence and this need not be the case. Select a prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$ such that $\mathbf{H}(\mathfrak{q})$ is torsion free and note $\mathbf{H}(\mathfrak{q})$ and $\pi_1(M)$ are commensurable. By Theorem 7.3, the group $\mathbf{H}(\mathfrak{q})$ admits an unbounded discrete isospectral deformation with

$$\left\{r_{\mathfrak{b}_i}, \mathbf{H}(\mathcal{O}_F/\mathfrak{b}_i),\ {}^{\mathbf{H}}\mathcal{AC}(\mathcal{O}_F/\mathfrak{b}_i)\right\}_{\mathcal{I}(\mathcal{P}(\mathbf{H}(\mathfrak{q})))}$$

By Proposition 4.2 (b) applied to the manifold M' associated $\mathbf{H}(\mathfrak{q})$, we obtain the desired isospectral covers. $\square$

8.2 Some examples For concreteness, we construct manifolds for which Theorem 8.2 is applicable. This section is also intended to illustrate how generic the assumption on the field of definition is.

1. Hyperbolic 3–manifolds We refer the reader to [21] for the definition of the invariant trace field of a hyperbolic 3–manifold. The following is an immediate corollary to Theorem 8.2—the constants come from (5) and (6).

Corollary 8.3. *Let M be closed hyperbolic 3–manifold. If the invariant trace field k_M of M satisfies $[k_M : \mathbf{Q}] \geq 14$, then M is commensurable with a manifold possessing arbitrarily large sets of isospectral, nonisometric covers. In addition, if M is nonarithmetic, we only require $[k_M : \mathbf{Q}] \geq 7$.*

Numerous examples satisfying Corollary 8.3 are produced as follows. By work of Thurston, for a hyperbolic knot complement M, performing $1/n$ surgery on the torus boundary produces closed hyperbolic 3–manifolds whose invariant trace field has arbitrarily large degree—see [16] for the last statement. In addition, for large n these manifolds will typically be nonarithmetic.

2. Manifolds modelled on unitary groups By a *CM field* E/F, we mean a quadratic extension E/F with E totally imaginary and F totally real. For concreteness, we take $E = \mathbf{Q}(\zeta_n)$ and $F = \mathbf{Q}(\cos(2\pi/n))$ for which the degree $E/\mathbf{Q}$ is $\phi(n)$, where ϕ is the Euler function. There are precisely $\phi(n)/2$ complex embeddings of $\mathbf{Q}(\zeta_n)$ up to post-composition by automorphisms of $\mathbf{C}$ and we list these embeddings by $\tau_1, \dots, \tau_{\phi(n)/2}$. Associated to each τ_j is an associated real embedding σ_j of $\mathbf{Q}(\cos(2\pi/n))$ and these produce all of the distinct real embeddings of $\mathbf{Q}(\cos(2\pi/n))$. By the Weak Approximation Theorem, there exist $\alpha_1, \dots, \alpha_s \in F^\times$ such that $\sigma_j(\alpha_k) < 0$ if and only if $j = 1$. We form the hermitian matrix on E^{s+t} by

$$h = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_s & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}$$

For each embedding τ_j of E, we define the special unitary group associated to ${}^{\tau_j}h$ (the matrix obtained by applying τ_j to each of the coefficients) by

$$\mathrm{SU}({}^{\tau_j}h) = \left\{ x \in \mathrm{SL}(s+t;\mathbf{C}) \ : \ ({}^{\tau_j}h)^{-1} x^* \, {}^{\tau_j}h x = 1 \right\}$$

where $*$ is complex transposition. By our selection of $\alpha_1, \dots, \alpha_s$,

$$\mathrm{SU}({}^{\tau_j}h) \cong \begin{cases} \mathrm{SU}(t,s), & j = 1 \\ \mathrm{SU}(s+t), & \text{otherwise.} \end{cases}$$

In particular, $\mathrm{SU}({}^{\tau_1}h; \mathcal{O}_E)$ is an arithmetic lattice with associated F–form $\mathrm{SU}({}^{\tau_1}h)$. As the degrees of these fields become arbitrarily large, Theorem 8.2 is eventually applicable and produces isospectral examples of manifolds modelled on the unitary groups $\mathrm{SU}(t,s)$.

For $s = 1$, the manifolds associated to the lattices in the above discussion are complex hyperbolic t–manifolds and are called *first type*. In addition, when $t = 2$ and $s = 1$, we only require that $\phi(n) \geq 12$. For lattices arising from division algebras equipped with an involution of second kind, the above fields can also be used and for $t = 2$ and $s = 1$, we have the identical condition $\phi(n) \geq 12$ needed in order to apply Theorem 8.2. These lattices and their associated manifolds are called *second type* and in the case $t = 2$ and $s = 1$, first and second type manifolds span the collection of arithmetic manifolds.

One class of manifolds for which Theorem 8.2 is not applicable are those containing fake projective planes. According to recent work of Prasad–Yeung [32], the field of definition never satisfies (5).

3. Quaternionic Shimura varieties For our last class of examples, we consider quaternionic Shimura varieties which are arithmetic manifolds modelled on the r–fold product of $\mathrm{PSL}(2;\mathbf{R})$.

Let F be a totally real number field with $s \geq r$ distinct real embeddings. Every F–defined quaternion algebra B is described by its associated Hilbert symbol

$$\left(\frac{\alpha, \beta}{F} \right)$$

where $\alpha, \beta \in F^\times$. The algebra B is the quotient of the free F–algebra $F[x,y]$ by the ideal $(x^2 - \beta, y^2 - \beta, xy + yx)$. For each embedding σ_j of F, we obtain a new algebra ${}^{\sigma_j}B$ whose Hilbert symbol is given by

$$\left(\frac{\sigma_j(\alpha), \sigma_j(\beta)}{\sigma_j(F)} \right).$$

We say B is admissible if there exist $\sigma_{j_1}, \dots, \sigma_{j_r}$ such that

$${}^{\sigma_j}B \otimes_{\sigma_j(F)} \mathbf{R} \cong \begin{cases} \mathrm{M}(2;\mathbf{R}), & \text{if } j = j_i,\ i = 1, \dots, r \\ \mathbb{H}, & \text{otherwise.} \end{cases}$$

For an admissible algebra B and an $\mathcal{O}_F$–order $\mathcal{O}$, $P\mathcal{O}^1$ is an arithmetic lattice in the r–fold product of $\mathrm{PSL}(2;\mathbf{R})$. Theorem 8.2 produces isospectral covers of the associated manifold if $[F:\mathbf{Q}] \geq 8$. When $r \geq 8$, this is automatically satisfied. Note, the existence of infinitely many isomorphism types of admissible F–quaternion algebras is a consequence of the Albert–Hasse–Brauer–Noether Theorem.

As mentioned in the introduction, Vigérnas [41] constructed examples of quaternionic Shimura varieties which are isospectral and non-isometric. As a consequence of the above discussion, we see that Theorem 8.2 provides many examples beyond those constructed in [41].

Corollary 8.4. *Let M be a quaternionic Shimura variety modelled on $(\mathrm{PSL}(2;\mathbf{R}))^r$ with $r \geq 8$. Then M satisfies the conditions of Theorem 8.2. In particular,* ***every*** *commensurable class contains isospectral, nonisometric manifolds.*

9 Super polynomial growth

Recall for a symmetric space X whose isometry group G has finite center and no compact factors, $\mathrm{SD}_X(t)$ is the maximum number of isometry classes of manifolds M_j modelled on X that are pairwise isospectral and have $\mathrm{vol}(M_j) \leq t$. For $\mathrm{N}(g)$, the maximum number of isospectral, nonisometric surfaces of genus g, Buser [8] established the upper bound

$$\mathrm{N}(g) \leq e^{720g^2}.$$

More generally, by a theorem of Wang [42], for $X \neq \mathbf{H}^2$ or $\mathbf{H}^3$, the number of manifolds M modelled on X with volume less than or equal to t is finite, and we denote this number by $a_X(t)$. To our knowledge, $a_X(t)$ is the only general upper bound for $\mathrm{SD}_X(t)$.

Brooks, Gornet, and Gustafson [7] derived the first super-polynomial lower bound for $\mathrm{N}(g)$, producing a sequence of g_i and a constant D for which

$$\mathrm{N}(g_i) \geq g_i^{D\log(g_i)}.$$

In this section, we establish some nontrivial lower bounds for $\mathrm{SD}_X(t)$, which to our knowledge are the first nontrivial ones in general.

9.1 Arithmetic growth As most of the Lie groups $\mathbf{G}$ considered here have property T—see [17]—no lattice in $\mathbf{G}$ can virtually surject a free group. Moreover, when a lattice Γ has the congruence subgroup property—see [18] and [31, Ch. 8]—the number of finite index subgroups of Γ of index at most t is proportionate logarithmically to $t^{\frac{\log t}{\log\log t}}$—see [10]. In particular, bounds similar to those of Brooks–Gornet–Gustafson are inaccessible by our methods. Recently, using Golod–Shafarevich towers Belolipetsky and Lubotzky [3] achieved $t^{\log t}$ asymptotic growth for the number of X–manifolds of volume no more than t. However, this construction cannot be utilized to establish upper bounds for $\mathrm{SD}_X(t)$ without producing incommensurable isospectral manifolds.

Proof of Theorem 1.3. Take a sequence F_j of fields with associated admissible F_j–forms $\mathbf{H}_j$ such that $n_j = [F_j : \mathbf{Q}]$ is monotonically increasing and with each F_j satisfying (5). By Theorem 7.3, there exists a discrete isospectral deformation of $\mathbf{H}_j(\mathcal{O}_{F_j})$ with

$$\left\{r_{\mathfrak{b}_i}, \mathbf{H}_j(\mathcal{O}_{F_j}/\mathfrak{b}_i),\ {}^{\mathbf{H}_j}\mathcal{AC}(\mathcal{O}_{F_j}/\mathfrak{b}_i)\right\}_{\mathcal{I}(\mathcal{P}(\Lambda_j))}$$

for some ideal set $\mathcal{P}(\Lambda_j)$. To produce the desired asymptotic growth, we will select among the infinitude of prime ideals in $\mathcal{P}(\Lambda_j)$, an ideal $\mathfrak{p}_{\ell,j}$, and only consider the number of isospectral, nonisometric manifolds obtained from pullbacks of twisted horizontal subgroups of $\mathfrak{N}_3(\mathcal{O}_{F_j}/\mathfrak{p}_{\ell,j})$. For each prime ideal $\mathfrak{p}_{\ell,j} \in \mathcal{P}(\Lambda_j)$, the local degree of the finite field extension $[\mathcal{O}_{F_j}/\mathfrak{p}_{\ell,j} : \mathbb{F}_{p_{\ell,j}}]$ is n_j and the number of distinct pullbacks of twisted horizontal groups is at least

$$(C'')^{-1} p_{\ell,j}^{n_j^2 - n_j(1+2\dim\mathbf{G})}.$$

If C_j is the volume of the arithmetic orbifold M_j associated to $\mathbf{H}_j(\mathcal{O}_{F_j})$, we can select $\mathfrak{p}_{\ell,j} \in \mathcal{P}(\Lambda_j)$ so that $C_j = p_{j,\ell}^{\varepsilon}$ with $\varepsilon > 0$ arbitrarily small. The volume of the manifolds $M_{j,\ell,k}$ (which vary over some index k) associated to the pullbacks of the twisted horizontal subgroups are

$$\begin{aligned}\mathrm{vol}(M_{j,\ell,k}) &= C[\Lambda_j : \pi_1(M_{j,\ell,k})] \\ &\leq C p_{\ell,j}^{n_j \dim\mathbf{G}} \\ &= p_{\ell,j}^{n_j \dim\mathbf{G} + \varepsilon}.\end{aligned}$$

Set r_j to be the largest positive integer such that

$$n_j^2 - n_j(1 + (2 + r_j)\dim\mathbf{G}) > 2$$

and select ε such that $r_j\varepsilon < 1$; this can be done since r_j is independent of the index

ℓ in p_ℓ. Then we have

$$\begin{aligned}
\frac{\mathrm{SD}_X(\mathrm{vol}(M_{j,\ell,k}))}{\mathrm{vol}(M_{j,\ell,k})^{r_j}} &\geq \frac{\mathrm{SD}_X(\mathrm{vol}(M_{j,\ell,k}))}{\left(p_{\ell,j}^{n_j \dim \mathbf{G}+\varepsilon}\right)^{r_j}} \\
&\geq \frac{p_{\ell,j}^{n_j^2-n_j(1+2\dim \mathbf{G})}}{C'' p_{\ell,j}^{r_j n_j \dim \mathbf{G}+r_j\varepsilon}} \\
&= (C'')^{-1} p_{\ell,j}^{n_j^2-n_j(1+(2+r_j)\dim \mathbf{G})-r_j\varepsilon} \\
&\geq (C'')^{-1} p_{\ell,j}^{2-r_j\varepsilon} \\
&\geq (C'')^{-1} p_{\ell,j}.
\end{aligned}$$

As n_j is monotonically increasing, the sequence r_j is monotonically increasing. In addition, we can select $p_{\ell,j}$ so that both

$$\mathrm{vol}(M_{j,\ell,k}) < \mathrm{vol}(M_{j+1,\ell,k}) \text{ and } (C'')^{-1} p_{\ell,j} \geq 1,$$

are satisfied. Setting $t_j = \mathrm{vol}(M_{j,\ell,k})$, we have increasing sequences $\{t_j\}$ and $\{r_j\}$ such that $\mathrm{SD}_X(t_j) \geq t_j^{r_j}$. □

Remark. We point out that in the proof we only produce super-polynomial growth by varying over an infinite number of wide commensurability classes.

9.2 Nonarithmetic growth For large nonarithmetic groups, we can produce identical asymptotic growth for $\mathrm{SD}_X(t)$ as achieved by Brooks, Gornet, and Gustafson. Namely, Theorem 1.4 of the introduction which we now prove.

Proof of Theorem 1.4. Let Λ be a large cocompact lattice in the isometry group $\mathbf{G}$ of X. For every p, the groups $\mathrm{SL}(3;\mathbb{F}_p[x])$ are finitely generated and so for any p, there exists a finite index subgroup Λ_0 of Λ and a surjective homomorphism

$$\rho\colon \Lambda_0 \longrightarrow \mathrm{SL}(3;\mathbb{F}_p[x]).$$

For each j, we have a surjective homomorphism

$$r_j\colon\ \mathrm{SL}(3;\mathbb{F}_p[x]) \longrightarrow \mathrm{SL}(3;\mathbb{F}_p[x]/x^j).$$

The group $\mathrm{SL}(3;\mathbb{F}_p[x]/x^j)$ contains the Heisenberg group $\mathfrak{N}_3(\mathbb{F}_p[x]/x^j)$. According to [7], there are

$$p^{j(j-1)/2}$$

$\mathfrak{N}_3(\mathbb{F}_p[x]/x^j)$–conjugacy classes of twisted horizontal subgroups ${}^f H(\mathbb{F}_p[x]/x^j)$. Identifying those which are conjugate in $\mathrm{SL}(3;\mathbb{F}_p[x]/x^j)$, we obtain at least

$$p^{j(j-1)/2-9j}$$

Λ_0–conjugacy classes of pullbacks of twisted horizontal subgroups. As Λ is nonarithmetic, there exists a constant D_p such that there are at least

$$p^{j(j-1)/2-9j-D_p}$$

distinct pullbacks of twisted horizontal subgroups—the constant D_p depends on $[\mathrm{Comm}_{\mathrm{Isom}(\mathbf{G}/\mathbf{K})}(\Lambda):\Lambda]$. For fixed p, as j increasing this number becomes unbounded.

For a fixed prime p and each j, let C_j denote the volume of a manifold $M_{j,k}$ associated to a twisted horizontal subgroup over $\mathbb{F}_p[x]/x^j$. Then

$$C_j = C[\Lambda : \pi_1(M_{j,k})] \leq Cp^{9j} \leq p^{9j+\delta}$$

where C is the volume of the orbifold associated to Λ and δ is given by $p^\delta = C$. With this, we have

$$\begin{aligned}
\mathrm{vol}(M_{j,k})^{\log(\mathrm{vol}(M_{j,k}))} &\leq \left(p^{9j+\delta}\right)^{\log(p^{9j+\delta})} \\
&= p^{(9j+\delta)^2 \log p} \\
&= p^{(81j^2+18\delta j+\delta^2)\log p}.
\end{aligned}$$

Letting $D = \frac{1}{324 \log p}$, we have

$$\begin{aligned}
\frac{\mathrm{SD}_X(\mathrm{vol}(M_{j,k}))}{\mathrm{vol}(M_{j,k})^{D\log(\mathrm{vol}(M_{j,k}))}} &\geq \frac{p^{\frac{j^2}{2}-\frac{19}{2}j-D_p}}{p^{D(81j^2+18\delta j+\delta^2)\log p}} \\
&= p^{\frac{j^2}{4}-\left(\frac{171+\delta}{18}\right)j-(D_p+\delta^2)}.
\end{aligned}$$

For sufficiently large j,

$$\frac{j^2}{4} - \left(\frac{171+\delta}{18}\right) j - (D_p + \delta^2) > 1,$$

and thus for such j

$$\mathrm{SD}_X(M_{j,k}) \geq \mathrm{vol}(M_{j,k})^{D\log(M_{j,k})}.$$

Setting $t_j = \mathrm{vol}(M_{j,k})$ produces the desired sequence with D so that

$$\mathrm{SD}_X(t_j) \geq t_j^{D\log t_j}.$$

□

Taking $\mathrm{SL}(2;\mathbb{F}_p[x])$ instead, we can get the better constant $\frac{1}{8\log 2}$; the best constant from the above argument is $\frac{1}{162\log 2}$.

10 Constructing isospectral towers

We have worked with ideal sets given by products of prime ideals despite the fact that the individual prime ideals suffice for the construction of arbitrarily large collections of isospectral covers. The reason is our need for an inverse limit system for the associated isospectral deformation in the construction of isospectral towers. This choice will now be used to prove the main result of this section. Namely, following theorem which implies Theorem 1.5.

Theorem 10.1. *Let M be a closed, irreducible, locally symmetric manifold modelled on a semisimple Lie group $\mathbf{G}$ with finite center and no compact factors. If the trace field of $\pi_1(M)$ satisfying (5), then M is commensurable with a manifold M' that possesses an isospectral tower*

Proof. For brevity, we only treat the arithmetic case. For an arithmetic M, there exists an admissible F–form $\mathbf{H}$ of $\mathbf{G}$ with F such that $\pi_1(M)$ and $\mathbf{H}(\mathcal{O}_F)$ are commensurable. By Theorem 7.3, $\mathbf{H}(\mathcal{O}_F)$ admits an unbounded discrete isospectral deformation with

$$\left\{r_{\mathfrak{b}_i}, \mathbf{H}(\mathcal{O}_F/\mathfrak{b}_i),\ {}^{\mathbf{H}}\mathcal{AC}(\mathcal{O}_F/\mathfrak{b}_i)\right\}_{\mathcal{P}(\mathbf{H}(\mathcal{O}_F))}$$

for some infinite prime ideal set $\mathcal{P}(\mathbf{H}(\mathcal{O}_F))$. By construction, we have natural maps

$$\mathbf{H}(\mathcal{O}_F/\mathfrak{b}_{i+1}) \longrightarrow \mathbf{H}(\mathcal{O}_F/\mathfrak{b}_i)$$

which restrict to maps

$$\mathfrak{N}_3(\mathcal{O}_F/\mathfrak{b}_{i+1}) \longrightarrow \mathfrak{N}_3(\mathcal{O}_F/\mathfrak{b}_i).$$

A twisted horizontal subgroup H of the form

$$H = \prod_{\ell=1}^{i+1} H_\ell$$

under this map has image

$$\prod_{\ell=1}^{i} H_\ell.$$

The number of $\mathbf{H}(\mathcal{O}_F/\mathfrak{b}_{i+1})$–distinct conjugacy classes of twisted horizontal subgroups of $\mathfrak{N}_3(\mathcal{O}_F/\mathfrak{b}_{i+1})$ which map to this subgroup is at least

$$p_{i+1}^{n_{i+1}^2 - n_i(1+\dim \mathbf{H})}.$$

As the number of twisted horizontal subgroups with non-isomorphic pullbacks to $\mathbf{H}(\mathcal{O}_F)$ is at least

$$(C'')^{-1} \prod_{\ell=1}^{i} p_\ell^{n_\ell^2 - (1+2\dim \mathbf{H})},$$

for sufficiently large i, there exist a pair of twisted horizontal subgroups $H_{i,1}$ and $H_{i,2}$ whose pullback are non-isomorphic. It suffices to find a pair of twisted horizontal subgroups $H_{k,1}$ and $H_{k,2}$ of $\mathfrak{N}_3(\mathcal{O}_F/\mathfrak{b}_k)$ which map to $H_{i,1}$ and $H_{i,2}$, respectively, under the map

$$\mathfrak{N}_3(\mathcal{O}_F/\mathfrak{b}_k) \longrightarrow \mathfrak{N}_3(\mathcal{O}_F/\mathfrak{b}_{k-1}) \longrightarrow \dots \longrightarrow \mathfrak{N}_3(\mathcal{O}_F/\mathfrak{b}_{i+1}) \longrightarrow \mathfrak{N}_3(\mathcal{O}_F/\mathfrak{b}_i)$$

and have non-isomorphic pullbacks to $\mathbf{H}(\mathcal{O}_F)$. For each $k > i$, let $\mathcal{AC}_{k,*}$ denote the set of $\mathbf{H}(\mathcal{O}_F/\mathfrak{b}_k)$–distinct twisted horizontal subgroups of $\mathfrak{N}_3(\mathcal{O}_F/\mathfrak{b}_k)$ which map to $H_{i,*}$ for $* = 1,2$. It follows that

$$|\mathcal{AC}_{k,*}| \geq \prod_{\ell=i+1}^{k} p_\ell^{n_\ell^2 - n_\ell(1+\dim \mathbf{H})}.$$

For a random $H_{k,1}$ and $H_{k,2}$ in $\mathcal{AC}_{k,1}$ and $\mathcal{AC}_{k,2}$, the probability that $H_{k,1}$ and $H_{k,2}$ have isomorphic pullbacks in $\mathbf{H}(\mathcal{O}_F)$ will be denoted by $\mu(H_{k,1},H_{k,2})$. By Theorem 3.3, we have

$$\begin{aligned}
\mu(H_{k,1},H_{k,2}) &\leq \min\left\{ \frac{C'' \prod_{\ell=1}^{k} p_\ell^{n_\ell \dim \mathbf{H}}}{|\mathcal{AC}_1| \cdot |\mathcal{AC}_2|}, 1 \right\} \\
&\leq \min\left\{ \left(C'' \prod_{\ell=1}^{k} p_\ell^{n_\ell \dim \mathbf{H}} \right) \left(\prod_{\ell'=i+1}^{k} p_{\ell'}^{n_{\ell'}^2 - n_{\ell'}(1+\dim \mathbf{H})} \right)^2, 1 \right\} \\
&= \min\left\{ \left(\frac{C''}{2} \prod_{\ell=1}^{i} p_\ell^{n_\ell \dim \mathbf{H}} \right) \prod_{\ell=i+1}^{k} p_\ell^{2n_\ell(1+2\dim \mathbf{H}) - 2n_\ell^2}, 1 \right\}.
\end{aligned}$$

As F satisfies (5) and by our selection of the ideals $\mathfrak{p}_\ell$,

$$n_\ell(1+2\dim \mathbf{H}) - n_\ell^2 < -1$$

for all n_ℓ. Therefore

$$\mu(H_{k,1},H_{k,2}) \leq \min\left\{ \left(C'' \prod_{\ell=1}^{i} p_\ell^{n_\ell \dim \mathbf{H}} \right) \prod_{\ell=i+1}^{k} p_\ell^{-2}, 1 \right\}.$$

As

$$C'' \prod_{\ell=1}^{i} p_\ell^{n_\ell \dim \mathbf{H}}$$

is constant, for any $\varepsilon << 1$, we can take k sufficiently large such that

$$\left(C'' \prod_{\ell=1}^{i} p_\ell^{n_\ell \dim \mathbf{H}} \right) \prod_{\ell=i+1}^{k} p_\ell^{-2} < \varepsilon.$$

Hence,

$$\mu(H_{k,1},H_{k,2}) < \varepsilon$$

for large k and so there exists a pair of twisted horizontal subgroups $H_{k,1}$ and $H_{k,2}$ which map to $H_{i,1}$ and $H_{i,2}$, respectively, having non-isomorphic pullbacks in $\mathbf{H}(\mathcal{O}_F)$.

From this, we get two pairs of infinite sequences $\{H_{i_\ell,1}\}_\ell$ and $\{H_{i_\ell,2}\}$ of twisted horizontal subgroups in $\mathbf{H}(\mathcal{O}_F/\mathfrak{b}_{i_\ell})$ with non-isomorphic pullbacks to $\mathbf{H}(\mathcal{O}_F)$. By Proposition 8.1, we may safely assume the pullback subgroups are torsion free and thus this produces the desired towers of isospectral, nonisometric manifolds. □

Given Corollary 8.3, we again can conclude that a generic closed hyperbolic 3–manifold satisfies the conditions of the above theorem. In addition, given Corollary 8.4, for quaternionic Shimura modular varieties of sufficiently high dimension, Theorem 10.1 is applicable. This shows that our methods provide more isospectral towers than just those given by Vigérnas [41].

11 An application in number theory

For a number field k and a nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_k$,

$$N(\mathfrak{a}) = |\mathcal{O}_k/\mathfrak{a}|$$

is the *norm* of $\mathfrak{a}$. The ζ*–function* for $k/\mathbf{Q}$ is defined by

$$\zeta_k(s) = \sum_{\mathfrak{a}} (N(\mathfrak{a}))^{-s}$$

for $s \in \mathbf{C}$. The important result required in the proof of Theorem 1.6 is the following (see Perlis [27]).

Theorem 11.1 (Perlis)**.** *Let $k/\mathbf{Q}$ be a finite Galois extension with Galois group N and H_1, H_2 almost conjugate subgroups of N. If $k_1 = \mathrm{Fix}(H_1)$ and $k_2 = \mathrm{Fix}(H_2)$, then*

$$\zeta_{k_1}(s) = \zeta_{k_2}(s).$$

Proof of Theorem 1.6. Let $\mathcal{P}$ be an infinite set of prime integers. By [36], there exists a Galois extension L_j with Galois group $\mathfrak{N}_3(\mathbf{Z}/p_j\mathbf{Z})$ for each p_j in $\mathcal{P}$. For distinct primes p_j and $p_{j'}$, as $[L_j : \mathbf{Q}]$ and $[L_{j'} : \mathbf{Q}]$ are relatively prime, the composite field $L_jL_{j'}$ is Galois with Galois group

$$\mathfrak{N}_3(\mathbf{Z}/p_j\mathbf{Z}) \times \mathfrak{N}_3(\mathbf{Z}/p_{j'}\mathbf{Z}).$$

In particular, this produces an inverse limit system of fields L_i with Galois groups $\mathfrak{N}_3(\mathbf{Z}/b_i\mathbf{Z})$, where

$$b_i = \prod_{\ell=1}^{i} p_\ell.$$

As in the proof of Theorem 10.1, we can find two towers of finite extensions $\{k_i\}$ and $\{k_i'\}$ whose corresponding Galois groups form two families of distinct twisted horizontal subgroups $\left\{ {}^{f_i}H(\mathbf{Z}/b_i\mathbf{Z})\right\}$ and $\{ {}^{g_i}H(\mathbf{Z}/b_i\mathbf{Z})\}$ that are almost conjugate and nonconjugate for all i. By our selection of f_i and g_i, the fields are not isomorphic. The proof is completed with an application of Theorem 11.1. □

12 Final remarks

12.1 Isospectral siblings and volume growth For a closed manifold M, we say M has an *isospectral sibling* if there exists a closed manifold M' such that M and M' are isospectral and nonisometric. For any real positive t, let $\mathrm{I}_X(t)$ denote the number of isometry classes closed locally symmetric manifolds modelled on X possessing an isospectral sibling with $\mathrm{vol}(M) \le t$. As $\mathrm{SD}_X(t) \le \mathrm{I}_X(t)$, Theorem 1.3 shows $\mathrm{I}_X(t)$ is super-polynomial. Using deformations in other subgroups of the Borel subgroup $\mathbf{B}$ of $\mathbf{G}$, it might be possible to show

$$\sup_t \frac{\mathrm{I}_X(t)}{a_X(t)} > 0,$$

where $a_X(t)$ is the number of isometry classes of X–manifolds of volume no more than t. The counting arguments established by Lubotzky et al. (see for instance [20]) show the number of pullbacks of the subgroups of $\mathbf{B}(\mathbb{F}_q)$ produces the same asymptotic behavior as the number of congruence subgroups. When the arithmetic lattice Λ of $\mathbf{G}$ has the congruence subgroup property, this provides some weight to this. However, as noted in [7], it is much more difficult to produce the same wealth of almost conjugate subgroups in other nilpotent groups residing in $\mathbf{B}(\mathbb{F}_q)$. Moreover, most of the pullbacks used in the count for $\mathbf{B}(\mathbb{F}_q)$ are subgroups of the torus $\mathbf{T}(\mathbb{F}_q)$ in $\mathbf{B}(\mathbb{F}_q)$ and so we would be forced to work with solvable groups.

12.2 Arbitrarily large families of towers The same style of argument used in the tower construction yields the following theorem.

Theorem 12.1. *Let M be a closed, irreducible, locally symmetric manifold modelled on a semisimple Lie group with finite center and no compact factors. If the field of definition for M satisfies (5), then M is commensurable with a manifold M' such that for every n, there exists n infinite towers of finite covers $M_j^1, \dots, M_j^n$ of M' and each j, the covers $M_j^1, \dots, M_j^n$ are pairwise isospectral and nonisometric.*

A similar generalization of Theorem 1.6 is valid too.

Theorem 12.2. *For every $\ell > 1$, there exists ℓ towers of finite extensions $k_j^1, \dots, k_j^\ell$ of $\mathbf{Q}$ such that for all $j \in \mathbf{N}$ and all distinct i, i', $k_j^i \neq k_j^{i'}$ and*

$$\zeta_{k_j^i}(s) = \zeta_{k_j^{i'}}(s).$$

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