Aspherical fully residually free Kähler groups are Fuchsian

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Abstract

In this note, we prove that an aspherical fully residually free Kähler group is Fuchsian. One instances of this shows that cocompact lattices in $\mathrm{PU}(1,n)$ for $n > 1$ cannot be residually free. The proof uses some recent work of Delzant and Gromov.

1 Introduction

Let M be a closed hyperbolic 3–manifold. Thurston asked whether or not M has a finite cover $\widetilde{M}$ such that $\widetilde{M}$ is fibered. A positive answer to this question would imply that every closed hyperbolic 3–manifold has a finite cover with positive first Betti number.

One approach to construct a counterexample to this question is as follows. A group G is called **fully residually free** if for each finite set $g_1, \dots, g_r \in g$, $g_j \neq 1$, there exists a surjective homomorphism $\rho\colon G \longrightarrow F_n$ such that $\rho(g_j) \neq 1$. It we insist this only holds for $r = 1$, then G is called **residually free**. By the work of Baumslag [1], a residually free, non-fully residually free group contains a subgroup H isomorphic to $F_2 \times \mathbf{Z}$. In particular, for word hyperbolic group, residually free and fully residually free are equivalent. By the work of I. Kapovich [3, 4] a fully residually free word hyperbolic group has the following property: every finitely generated subgroup is geometrically finite. As a result, a closed hyperbolic 3–manifold group which is fully residually free cannot be virtually fibered, since the fiber group is not geometrically finite.

It is a difficult venture to construct fully residually free closed 3–manifold groups. However, since finitely generated subgroups of fully residually free groups are fully residually free, one might try to construct a fully residually free group Γ and an injective homomorphism $\rho\colon \pi_1(M) \longrightarrow \Gamma$. One class of groups for which injections might arise are closed complex hyperbolic n–orbifold groups. More generally, one can take the fundamental group of a closed Kähler manifold as a possible target group. We prove that such an injection is impossible.

Theorem 1.1. *Let Γ be the fundamental group of an aspherical, closed Kähler manifold. If Γ is non-abelian and fully residually free, then Γ has a finite index subgroup Γ_0 which is isomorphic to a surface group.*

2 Cut kernels

2.1 Cut groups and cut kernels

For a finitely generated group G and subgroup H, we say that H is a **cut group** if for some generating set $\{g_1, \ldots, g_r\}$, the relative Cayley graph $X(G/H; g_1, \ldots, g_r)$ has more than one end. It follows that any conjugate of a cut group is a cut group and the intersection of cut groups is a cut group. Consequently, the intersection of all cut groups is a normal subgroup of G which is a cut group. We call this the **cut kernel** and denote it by $K(G)$.

Lemma 2.1. *Let $\rho\colon G \longrightarrow H$ be a surjective homomorphism of finitely generated groups. Then $\rho(K(G)) \subset K(H)$.*

Lemma 2.2. *Let Γ be a residually free group with cut kernel K. Then $K = 1$.*

Proof. It is well known that $K(F_n) = 1$, for any non-abelian free group. Select $g \in G \setminus \{1\}$. Since G is residually free, there exists a surjective homomorphism $\rho_g\colon G \longrightarrow F_m$ such that $\rho_g(g) \neq 1$. By Lemma 2.1, $\rho_g(K(G)) < K(F_m) = 1$. Thus $g \notin K(G)$ and so $K(G) = 1$. □

2.2 A theorem of Gromov-Delzant

Let M be a compact Kähler manifold with fundamental group Γ. Let K denote the cut kernel of Γ. The following theorem is due to Gromov and Delzant.

Theorem 2.3 (Gromov-Delzant). *Let Γ be a compact Kähler group with associated manifold M. Then there exists*

$$\begin{array}{ccc} \widehat{M} & \longrightarrow & W \\ {\scriptstyle finite}\big\downarrow & & \\ M & & \end{array}$$

where W is a flat torus bundle over a product of closed surfaces such that

$$1 \longrightarrow K \cap \pi_1(\widehat{M}) \longrightarrow \pi_1(\widehat{M}) \longrightarrow \pi_1(W).$$

3 Fully residually free Kähler groups

In this section, we investigate fully residually free aspherical Kähler groups. Our two main tools are Theorem 2.3 and a result of Bridson, Howie, Miller, and Short [2].

3.1 Subgroups of surface group products

We require the following result of Bridson, Howie, Miller, and Short [2].

Theorem 3.1 (Bridson-Howie-Miller-Short; [2]). *If G is contained in a product of r surface groups and $H^j(G;\mathbf{Z})$ is finitely generated for $j=1,\dots,r$, then there exists $G_0 < G$, a finite index subgroup, which is a product of surface groups (possibly with boundary).*

3.2 Proof of Theorem 1.1

Let Γ be an aspherical, fully residually free Kähler group. By Theorem 2.3, there exists a finite index subgroup Γ_0 and an injective homomorphism

$$\rho\colon \Gamma_0 \longrightarrow \pi_1(W)$$

where

$$1 \longrightarrow \mathbf{Z}^n \longrightarrow \pi_1(W) \longrightarrow \prod_{j=1}^{r} \pi_1(\Sigma_g) \longrightarrow 1.$$

We need a lemma.

Lemma 3.2. $\rho(\Gamma_0) \cap \mathbf{Z}^n = 1$.

Proof. Since Γ is fully residually free and $\Gamma_0 \subset \Gamma$, Γ_0 is fully residually free. The intersection of $\rho(\Gamma_0)$ with $\mathbf{Z}^n$ is a normal subgroup which is fully residually free. Therefore, $\rho(\Gamma_0)\cap\mathbf{Z}^n$ is trivial or isomorphic to $\mathbf{Z}$. If $\rho(\Gamma_0)\cap\mathbf{Z}^n$ is isomorphic to $\mathbf{Z}$ and $\rho(\Gamma_0)$ is not contained in $\mathbf{Z}^n$, then Γ_0 contains an infinite index, normal abelian subgroup, which is impossible. Thus $\rho(\Gamma_0) \cong \mathbf{Z}$. By assumption G is non-abelian, and so $\rho(\Gamma_0) \cap \mathbf{Z}^n = 1$. □

By Lemma 3.2, the projection of $\rho(\Gamma_0)$ onto the product of surface groups is faithful. By Theorem 3.1, there exists a finite index subgroup Γ_1 which is isomorphic to a product of surface groups (possibly with boundary).

Lemma 3.3. *Let*

$$G = \prod_{j=1}^{s} \pi_1(\Sigma_{g_j,n_j})$$

be fully residually free with $H^j(G;\mathbf{Z})$ finitely generated for $j=1,\dots,s$. Then there exists $k \in \{1,\dots,s\}$ such that

$$\pi_k\colon G \longrightarrow \pi_1(\Sigma_{g_k,n_k})$$

is faithful.

Proof. By Theorem 3.1, G has a finite index subgroup G_0 which is isomorphic to a product of surface groups with boundary. If more than one product exists, G_0 contains a subgroups of the form $F_2 \times F_2$, which is impossible by the work of Baumslag [1]. Thus, G_0 is a surface group and it follows then that $G \cong \pi_1(\Sigma_{g_k,n_k})$ for some k. □

The proof of Theorem 1.1 now follows from Lemma 3.3, since Γ_1 is isomorphic to a surface group.

Corollary 3.4. *Let Λ be a lattice in* $\mathrm{Isom}(\mathbf{H}^n_{\mathbf{C}})$ *and assume that Λ is fully residually free. Then Λ is commensurable with a surface group.*

Proof. If Λ is cocompact, then we can apply Theorem 1.1. Thus, we may assume that Λ is non-cocompact. If $n > 1$, then Λ contains a non-abelian nilpotent group, namely the Fitting subgroup of a maximal peripheral subgroup of a cusp. Since Λ is fully residually free, this group is fully residually free, which is impossible. □

References

[1] Benjamin Baumslag, *Residually free groups*, Proc. London Math. Soc. (3) **17** (1967), 402–418.

[2] Martin R. Bridson, James Howie, Charles F. Miller, III, and Hamish Short, *The subgroups of direct products of surface groups*, Geom. Dedicata **92** (2002), 95–103, Dedicated to John Stallings on the occasion of his 65th birthday.

[3] Ilya Kapovich, *Subgroup properties of fully residually free groups*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 335–362 (electronic).

[4] ______, *Erratum to: "Subgroup properties of fully residually free groups" Trans. Amer. Math. Soc.* **354** *(2002), no. 1, 335–362*, Trans. Amer. Math. Soc. **355** (2003), no. 3, 1295–1296 (electronic).