

Length equivalent hyperbolic manifolds

Christopher J. Leininger, D. B. McReynolds, and Alan W. Reid*

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1 Introduction

Let M be a Riemannian manifold. The *length spectrum* $\mathcal{L}(M)$ of M is the set of all lengths of closed geodesics in M counted with multiplicities. The *weak length spectrum* $L(M)$ of M is the set of all lengths of closed geodesics in M , ignoring multiplicities.

In the case when M is an orientable Riemannian manifold of negative curvature $\mathcal{L}(M)$ and $L(M)$ are related to the *eigenvalue spectrum* of M ; that is, the set $\mathcal{E}(M)$ of all eigenvalues of the Laplace–Beltrami operator acting on $L^2(M)$ counted with multiplicities. For example, in this setting it is known that $\mathcal{E}(M)$ determines $L(M)$ (see [3]). In the case of closed hyperbolic surfaces, the stronger statement that $\mathcal{E}(M)$ determines $\mathcal{L}(M)$ and vice-versa holds [5, 6]. A pair of manifolds M_1, M_2 are said to be *isospectral* (resp. *iso-length spectral*) if $\mathcal{E}(M_1) = \mathcal{E}(M_2)$ (resp. $\mathcal{L}(M_1) = \mathcal{L}(M_2)$). Geometric and topological constraints are forced on isospectral manifolds, for example the manifolds have the same volume [10], and so for surfaces the same genus.

In this paper we address the issue of how much information is lost by simply assuming that M_1 and M_2 are *length equivalent*, by which we mean $L(M_1) = L(M_2)$. Examples of non-compact arithmetic hyperbolic 2-manifolds that are length equivalent were constructed in Theorem 2 of [14] using the arithmetic structure. However, as far as the authors are aware, no examples of closed surfaces that are length equivalent are known. Our main result ameliorates this situation for hyperbolic surfaces and indeed for all finite volume hyperbolic m -manifolds.

Theorem 1.1. *Let M be a finite volume hyperbolic m -manifold. Then there exist infinitely many pairs of finite covers $\{M_j, N_j\}$ of M such that $L(M_j) = L(N_j)$ and*

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for which $\text{vol}(M_j)/\text{vol}(N_j) \rightarrow \infty$. Moreover, $L(M_j) = L(N_j)$ for any Riemannian metric on M .

Theorem 1.1 follows from a general construction involving only the fundamental group, see Theorem 2.1. In particular, the next result provides another class of examples. Recall that a group Γ is called *large* if Γ contains a finite index subgroup which admits an epimorphism onto a non-abelian free group.

Theorem 1.2. *Let M be a finite volume Riemannian manifold whose fundamental group is large. Then there exist infinitely many pairs of finite covers $\{M_j, N_j\}$ of M such that $L(M_j) = L(N_j)$ and for which $\text{vol}(M_j)/\text{vol}(N_j) \rightarrow \infty$.*

As will be visible from our construction of these covering spaces, the extension to manifolds admitting locally symmetric geometries modelled on non-compact simple Lie groups with finite center is straightforward. We discuss this in §5.

2 Sunada's method

Our construction is motivated by one of Sunada [17] (which in turn was motivated from number theory) for producing isospectral and iso-length spectral manifolds, and which we now recall.

For any finite group G and subgroups H and K of G , we say that (H, K) is an *almost conjugate pair* if for any g' in G the following condition holds:

$$|H \cap \{gg'g^{-1} : g \in G\}| = |K \cap \{gg'g^{-1} : g \in G\}|.$$

In [17] Sunada proved the following theorem relating almost conjugate pairs with isospectral covers.

Theorem (Sunada). *Let M be a closed Riemannian manifold, G a finite group, and (H, K) an almost conjugate pair in G . If $\pi_1(M)$ admits a homomorphism onto G , then the finite covers M_H and M_K associated to the pullback subgroups of H and K are isospectral. Moreover, the manifolds M_H and M_K are iso-length spectral.*

Checking that the manifolds produced by Sunada's method are non-isometric requires more work. However, for length equivalence far less is required, and the issue of non-isometric is built into our construction.

Definition: *For any group G and subgroups H and K of G , we say that (H, K) is a conjugacy covering pair if for any g' in G the following condition holds:*

$$H \cap \{gg'g^{-1} : g \in G\} \neq \emptyset \text{ if and only if } K \cap \{gg'g^{-1} : g \in G\} \neq \emptyset. \quad (1)$$

Note that it is immediate from the definition that an almost conjugate pair (H, K) is a conjugacy covering pair.

Theorem 2.1. *Let M be a Riemannian manifold, G a group, and (H, K) a conjugacy covering pair in G . If $\pi_1(M)$ admits a homomorphism onto G , then the covers M_H and M_K associated to the pullback subgroups of H and K are length equivalent.*

Proof. Because closed geodesics on M_H and M_K are precisely lifts of geodesics on M , to prove the Theorem, it suffices to show that a closed geodesic γ on M has a lift to a closed geodesic on M_H if and only if γ has a lift to a closed geodesic on M_K .

By standard covering space theory, we note that γ has a closed lift to M_H if and only if the associated element $[\gamma] \in \pi_1(M)$ is $\pi_1(M)$ -conjugate into $\pi_1(M_H)$. Thus, let γ be a closed geodesic which has a closed lift to M_H . By assumption (1), since $\pi_1(M_H)$ contains a conjugate of $[\gamma]$, then some conjugate of $[\gamma]$ is an element of $\pi_1(M_K)$. Hence, γ also has a lift to a closed geodesic on M_K . Switching the roles of H and K in the above argument shows a closed geodesic which has a lift to a closed geodesic on M_K also has a lift to a closed geodesic on M_H . \square

3 Conjugacy covering pairs

In this section we introduce two infinite families of finite groups which contain conjugacy covering pairs. Some additional terminology is required.

Let

$$1 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 1$$

be an extension with abelian kernel A . The *holonomy representation*

$$\varphi: H \longrightarrow \text{Aut}(A)$$

is obtained from the descent to H of the action of G on A by conjugation. Explicitly, since A is abelian, it acts trivially on itself, while the short exact sequence provides the isomorphism between G/A and H . The following lemma allows us to decide whether or not a pair in an extension G is a conjugacy covering pair.

Lemma 3.1. *A pair (A_1, A_2) in A is a conjugacy covering pair in G if and only if for any $a \in A$*

$$\varphi(H)(a) \cap A_1 \neq \emptyset \text{ if and only if } \varphi(H)(a) \cap A_2 \neq \emptyset.$$

Proof. According to (1), we must show that a G -conjugacy class intersects A_1 if and only if it intersects A_2 . Since A is a normal subgroup of G , any G -conjugacy class that intersects A must be contained in A . Therefore, it suffices to show that

for any $a \in A$, the orbit of a under G -conjugation is the same as the orbit under the action of H by φ . That this holds follows at once from the definition of the holonomy representation. \square

Example 1: Let \mathbb{F}_p be a finite field of prime order p , and let $n \geq 2$ be a positive integer. The n -dimensional \mathbb{F}_p -affine group is defined to be the semidirect product $\mathbb{F}_p^n \rtimes \mathrm{SL}(n; \mathbb{F}_p)$. This is the split extension of the abelian group \mathbb{F}_p^n by $\mathrm{SL}(n; \mathbb{F}_p)$, where the holonomy representation $\mathrm{SL}(n; \mathbb{F}_p) \rightarrow \mathrm{Aut}(\mathbb{F}_p^n) = \mathrm{GL}(n; \mathbb{F}_p)$ is the identity. We call any non-trivial \mathbb{F}_p -vector subspace V of \mathbb{F}_p^n a *translational subgroup* of $\mathbb{F}_p^n \rtimes \mathrm{SL}(n; \mathbb{F}_p)$.

Lemma 3.2. *Let V and W be any two translational subgroups of $\mathbb{F}_p^n \rtimes \mathrm{SL}(n; \mathbb{F}_p)$. Then (V, W) is a conjugacy covering pair in $\mathbb{F}_p^n \rtimes \mathrm{SL}(n; \mathbb{F}_p)$.*

Proof. Since $\mathrm{SL}(n; \mathbb{F}_p)$ acts transitively on 1-dimensional subspaces of \mathbb{F}_p^n , given a non-trivial element $v \in \mathbb{F}_p^n$, the $\mathrm{SL}(n; \mathbb{F}_p)$ -orbit of v intersects both V and W . The lemma now follows from Lemma 3.1. \square

Example 2: Our next example is an extension of the ideas in Example 1.

Let $p > 3$ be a prime and R be the ring $\mathbb{Z}/p^2\mathbb{Z}$ with maximal ideal $\mathfrak{m} = p\mathbb{Z}/p^2\mathbb{Z}$. The short exact sequence

$$1 \longrightarrow \mathfrak{m} \longrightarrow R \xrightarrow{r_{\mathfrak{m}}} \mathbb{F}_p \longrightarrow 1$$

induces the short exact sequence

$$1 \longrightarrow V_p \longrightarrow \mathrm{SL}(2; R) \xrightarrow{\tilde{r}_{\mathfrak{m}}} \mathrm{SL}(2; \mathbb{F}_p) \longrightarrow 1$$

where $V_p \cong \mathbb{F}_p^3$. In a similar fashion to Example 1, we aim to show

Proposition 3.3. *There exists a subgroup W of V_p such that (V_p, W) is a conjugacy covering pair in $\mathrm{SL}(2; R)$ and $\dim_{\mathbb{F}_p} W = 2$.*

Before commencing with the proof of Proposition 3.3 we need to recall further details about the $\mathrm{SL}(2; \mathbb{F}_p)$ -action on V_p .

In a mild abuse of notation, we let $\tilde{r}_{\mathfrak{m}}$ also denote the induced homomorphism of matrix algebras

$$\tilde{r}_{\mathfrak{m}}: \mathrm{M}(2; R) \longrightarrow \mathrm{M}(2; \mathbb{F}_p).$$

Any section s of $r_{\mathfrak{m}}$ induces a section \tilde{s} of $\tilde{r}_{\mathfrak{m}}$ by simply applying s to the coefficients of a matrix, and for concreteness, we take the section

$$s(x \bmod p) = x \bmod p^2$$

where $x \in \{0, 1, \dots, p-2, p-1\}$. The holonomy representation φ for the $\mathrm{SL}(2; \mathbb{F}_p)$ -action on V_p is then given by

$$\varphi(T)(v) = \tilde{s}(T)v\tilde{s}(T^{-1})$$

where $T \in \mathrm{SL}(2; \mathbb{F}_p)$ and $v \in V_p$.

The natural embedding of $\mathrm{SL}(2; \mathbb{F}_p)$ in the algebra $\mathrm{M}(2; \mathbb{F}_p)$ affords us with a description of Lie algebra $\mathfrak{sl}(2; \mathbb{F}_p)$ of $\mathrm{SL}(2; \mathbb{F}_p)$ as the Lie subalgebra of $\mathrm{M}(2; \mathbb{F}_p)$ of those matrices with trace zero. With this description, we define the map

$$\tau: \mathfrak{sl}(2; \mathbb{F}_p) \longrightarrow V_p$$

by

$$\tau(x) = I_2 + p\tilde{s}(x)$$

where I_2 is the identity matrix in $\mathrm{M}(2; R)$. Our next lemma provides an explicit description of the $\mathrm{SL}(2; \mathbb{F}_p)$ -action on V_p . In the statement of the lemma, the $\mathrm{SL}(2; \mathbb{F}_p)$ -action on $\mathfrak{sl}(2; \mathbb{F}_p)$ is the adjoint action given by conjugation.

Lemma 3.4. *τ is an isomorphism of both \mathbb{F}_p -vector spaces and $\mathrm{SL}(2; \mathbb{F}_p)$ -modules.*

Proof. To begin, for every $x, y \in \mathfrak{sl}(2; \mathbb{F}_p)$ and $T \in \mathrm{SL}(2; \mathbb{F}_p)$

$$\begin{aligned} p\tilde{s}(x+y) &= p\tilde{s}(x) + p\tilde{s}(y) \\ p(\tilde{s}(T)\tilde{s}(x)\tilde{s}(T)^{-1}) &= p\tilde{s}(TxT^{-1}), \end{aligned}$$

since when $a \equiv b \pmod{p}$, then $pa \equiv pb \pmod{p^2}$. That τ is a homomorphism is a result of the calculation

$$\begin{aligned} \tau(x)\tau(y) &= (I_2 + p\tilde{s}(x))(I_2 + p\tilde{s}(y)) = I_2 + p(\tilde{s}(x) + \tilde{s}(y)) + p^2\tilde{s}(x)\tilde{s}(y) \\ &= I_2 + p\tilde{s}(x+y) = \tau(x+y). \end{aligned}$$

Since this is visibly onto and these two groups have the same cardinality, one sees that this is an isomorphism of groups, and hence also an isomorphism of \mathbb{F}_p -vector spaces. In addition, for every $T \in \mathrm{SL}(2; \mathbb{F}_p)$ and $x \in \mathfrak{sl}(2; \mathbb{F}_p)$ we have

$$\begin{aligned} \varphi(T)(\tau(x)) &= \tilde{s}(T)(I_2 + p\tilde{s}(x))\tilde{s}(T)^{-1} = I_2 + \tilde{s}(T)p\tilde{s}(x)\tilde{s}(T)^{-1} \\ &= I_2 + p(\tilde{s}(T)\tilde{s}(x)\tilde{s}(T)^{-1}) = \tau(TxT^{-1}). \end{aligned}$$

Since the adjoint action of $\mathrm{SL}(2; \mathbb{F}_p)$ on $\mathfrak{sl}(2; \mathbb{F}_p)$ is given by conjugation, this shows that τ is also an $\mathrm{SL}(2; \mathbb{F}_p)$ -module isomorphism. \square

The vector space $\mathfrak{sl}(2; \mathbb{F}_p)$ is furnished with a natural $\mathrm{SL}(2; \mathbb{F}_p)$ -invariant bilinear form B , namely the Killing form. We denote the associated nondegenerate quadratic form by Q_B , its orthogonal group by $\mathrm{O}(Q_B; \mathbb{F}_p)$, and the commutator subgroup of $\mathrm{O}(Q_B; \mathbb{F}_p)$ by $\Omega(Q_B; \mathbb{F}_p)$ —see [4] for the necessary background on quadratic forms and their orthogonal groups. The invariance of Q_B under the adjoint action induces the homomorphism

$$\widehat{\Phi}: \mathrm{SL}(2; \mathbb{F}_p) \longrightarrow \mathrm{O}(Q_B; \mathbb{F}_p)$$

which upon factoring through the kernel of $\widehat{\Phi}$ supplies us with the injection

$$\Phi: \mathrm{PSL}(2; \mathbb{F}_p) \longrightarrow \mathrm{O}(Q_B; \mathbb{F}_p).$$

Since $\mathrm{PSL}(2; \mathbb{F}_p)$ is perfect for $p > 3$, it must be that $\Phi(\mathrm{PSL}(2; \mathbb{F}_p)) \subset \Omega(Q_B; \mathbb{F}_p)$. However, since these finite groups have the same cardinalities, these two groups must be equal.

The following lemma provides conjugacy covering pairs in this setting.

Lemma 3.5. *Let \mathfrak{W} be a 2-dimensional \mathbb{F}_p -vector subspace of $\mathfrak{sl}(2; \mathbb{F}_p)$ which contains a non-trivial Q_B -isotropic vector. Then for every $x \in \mathfrak{sl}(2; \mathbb{F}_p)$, there exists $S_x \in \Omega(Q_B; \mathbb{F}_p)$ such that $S_x x \in \mathfrak{W}$.*

Proof. Let $x \in \mathfrak{sl}(2; \mathbb{F}_p)$ be any nonzero vector. We must show the existence of an $S_x \in \Omega(Q_B; \mathbb{F}_p)$ such that $S_x x \in \mathfrak{W}$. We begin by first finding $S \in \mathrm{O}(Q_B; \mathbb{F}_p)$ with $Sx \in \mathfrak{W}$. To do this we argue as follows.

The restriction of Q_B to \mathfrak{W} endows \mathfrak{W} with a quadratic structure which, according to [15] p. 34 Proposition 4, represents every element of \mathbb{F}_p^* . That \mathfrak{W} contains a nontrivial Q_B -isotropic vector yields the immediate upgrade $Q_B(\mathfrak{W} \setminus \{0\}) = \mathbb{F}_p$, and hence we can find $y \in \mathfrak{W}$ such that $Q_B(x) = Q_B(y)$. For such a y , the \mathbb{F}_p -linear isomorphism φ of the line through x onto the line through y defined by $\varphi(x) = y$ is an isometry of the quadratic subspaces of $\mathfrak{sl}(2; \mathbb{F}_p)$. An application of Witt's theorem produces an extension $S \in \mathrm{O}(Q_B; \mathbb{F}_p)$ of φ with the desired property $Sx = y$.

If $S \in \Omega(Q_B; \mathbb{F}_p)$, we simply set $S_x = S$. To handle the case that $S \notin \Omega(Q_B; \mathbb{F}_p)$, recall that the spinor norm induces a homomorphism

$$\theta: \mathrm{O}(Q_B; \mathbb{F}_p) \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

for which $\ker \theta = \Omega(Q_B; \mathbb{F}_p)$. A complete set of coset representatives can be taken to be reflections τ_1 and τ_2 and their product τ_3 , where the reflections τ_1 and τ_2 can be chosen to be any reflections associated to vectors z_1 and z_2 for which $Q_B(z_1) = 1$,

$Q_B(z_2) = r$, and $r \in \mathbb{F}_p$ is any non-square. As noted above, Q_B restricted to \mathfrak{W} represents \mathbb{F}_p , and hence we can choose vectors $y_1, y_2 \in \mathfrak{W}$ so that the associated reflections τ_1 and τ_2 and their product τ_3 provide a complete set of coset representatives for $\Omega(Q_B; \mathbb{F}_p)$. As a reflection associated to an anisotropic vector preserves any subspace containing it, τ_i preserves \mathfrak{W} for each $i = 1, 2, 3$. The proof is completed by selecting τ_i such that $\theta(\tau_i)\theta(S)$ is trivial and setting $S_x = \tau_i S$. \square

Proof of Proposition 3.3. Let $\mathfrak{W} < \mathfrak{sl}(2; \mathbb{F}_p)$ be any 2-dimensional subspace containing a Q_B -isotropic vector and set $W = \tau(\mathfrak{W})$. The proposition will follow on showing that (V_p, W) is a conjugacy covering pair in $\mathrm{SL}(2; R)$. By Lemma 3.1 and the containment $W \subset V_p$, it suffices to show that for each $v \in V_p$, the $\mathrm{SL}(2; \mathbb{F}_p)$ -orbit of v under the holonomy action intersects W . According to Lemma 3.4, τ is both $\mathrm{SL}(2; \mathbb{F}_p)$ -equivariant and an isomorphism of \mathbb{F}_p -vector spaces, and so the aforementioned requirement is equivalent to showing that the $\mathrm{SL}(2; \mathbb{F}_p)$ -orbit of $x = \tau^{-1}(v)$ under the adjoint action intersects \mathfrak{W} . By Lemma 3.5, there exists $S_x \in \Omega(B_Q; \mathbb{F}_p)$ such that $S_x x \in \mathfrak{W}$. Since $\widehat{\Phi}(\mathrm{SL}(2; \mathbb{F}_p)) = \Omega(B_Q; \mathbb{F}_p)$, this yields an element $S \in \mathrm{SL}(2; \mathbb{F}_p)$ such that $\widehat{\Phi}(S)x \in \mathfrak{W}$. It now follows that the $\mathrm{SL}(2; \mathbb{F}_p)$ -orbit of x under the adjoint action intersects \mathfrak{W} , as needed. \square

4 Proofs of main results

Notice that Theorem 1.1 in the case of closed hyperbolic surfaces follows from Theorem 1.2. We defer the case of surfaces until then.

Proof of 1.1. Given the preceding comment, we shall assume that M is a finite volume hyperbolic manifold of dimension $m \geq 3$. We shall identify $\mathrm{Isom}(\mathbf{H}^m)$ with $\mathrm{PO}_0(m, 1)$ and let $\Gamma < \mathrm{PO}_0(m, 1)$ denote the fundamental group of M . We can assume that $\Gamma < \mathrm{PO}_0(m, 1; S)$ for a finite extension ring S of \mathcal{O}_k where k is the ring of fractions of S (see [12] for the details). The Theorem will follow upon producing surjections of Γ onto groups containing $\mathrm{PSL}(2; R)$ with R as in Example 2.

To this end, by Strong Approximation ([18] and [9]; see also [7] for a discussion of the proof in the particular case of hyperbolic manifolds) and the Cebotarev Density Theorem, there exists an infinite set of prime ideals \mathcal{P} of S such that $S/\mathfrak{p} = \mathbb{F}_p$ for all $\mathfrak{p} \in \mathcal{P}$, and for which the image of Γ under the reduction homomorphism

$$r_{\mathfrak{p}^j}: \mathrm{PO}_0(m, 1; S) \longrightarrow \mathrm{PO}(m, 1; S/\mathfrak{p}^j) = \mathrm{PO}(m, 1; \mathbb{Z}/p^j\mathbb{Z})$$

contains $\Omega(m, 1; \mathbb{Z}/p^j\mathbb{Z})$ for all $j \geq 1$. As above, $\Omega(m, 1; \mathbb{Z}/p^j\mathbb{Z})$ denotes the commutator subgroup of $\mathrm{O}(m, 1; \mathbb{Z}/p^j\mathbb{Z})$. We note that there is a natural inclusion of $\mathrm{PO}_0(m, 1; S)$ into $\mathrm{PO}_0(m, 1; S)$ given by the restriction of the standard signature

$(m, 1)$ form to the subspace spanned by e_{m-1}, e_m , and e_{m+1} . Restricting the homomorphism r_{p^j} to the subgroup $\mathrm{PO}_0(2, 1; S)$ yields subgroups $\mathrm{PO}(2, 1; \mathbb{Z}/p^j\mathbb{Z})$ in $\mathrm{PO}(m, 1; \mathbb{Z}/p^j\mathbb{Z})$. In particular, there are infinitely many primes p such that for all $j \geq 1$ we have

$$\Omega(2, 1; \mathbb{Z}/p^j\mathbb{Z}) < \Omega(m, 1; \mathbb{Z}/p^j\mathbb{Z}) < r_{p^j}(\Gamma) < \mathrm{PO}(m, 1; \mathbb{Z}/p^j\mathbb{Z}).$$

Specializing to $j = 2$, we see that $\Omega(2, 1; R) < r_{p^2}(\Gamma)$, where R is as in Example 2. That $r_{p^2}(\Gamma)$ contains a subgroup isomorphic to $\mathrm{PSL}(2; R)$ follows from the fact that the finite groups $\Omega(2, 1; \mathbb{Z}/p^j\mathbb{Z})$ and $\mathrm{PSL}(2; \mathbb{Z}/p^j\mathbb{Z})$ are isomorphic. To see the latter, first recall that the p -adic Lie groups $\mathrm{PSL}(2; \mathbb{Q}_p)$ and $\Omega(2, 1; \mathbb{Q}_p)$ are isomorphic (as p -adic Lie groups). The groups $\mathrm{PSL}(2; \mathbb{Z}_p)$ and $\Omega(2, 1; \mathbb{Z}_p)$ are the respective maximal compact subgroups of $\mathrm{PSL}(2; \mathbb{Q}_p)$ and $\Omega(2, 1; \mathbb{Q}_p)$, and are unique up to isomorphism (see [11, Ch 3.4]). Hence the groups $\mathrm{PSL}(2; \mathbb{Z}_p)$ and $\Omega(2, 1; \mathbb{Z}_p)$ are isomorphic as p -adic Lie groups. Reducing modulo the ideal generated by the j th power of the uniformizer π of \mathbb{Z}_p yields the asserted isomorphism between $\Omega(2, 1; \mathbb{Z}/p^j\mathbb{Z})$ and $\mathrm{PSL}(2; \mathbb{Z}/p^j\mathbb{Z})$. With this we see that $r_{p^2}(\Gamma)$ contains a subgroup isomorphic to $\mathrm{PSL}(2; R)$. Hence, by passage to a subgroup of finite index in Γ , we can arrange a finite cover X of M with

$$\pi_1(X) \twoheadrightarrow \mathrm{PSL}(2; R).$$

Visibly the kernel of the central quotient of $\mathrm{SL}(2; R)$ intersects V_p at the identity, and thus V_p fully survives the passage to $\mathrm{PSL}(2; R)$. With this, the existence of pairs $\{M_p, N_p\}_{p \in \mathcal{P}}$ as stated in Theorem 1.1 now follows from Theorem 2.1, Proposition 3.3, and the infinitude of \mathcal{P} ; the claim for volume ratios is verified by the fact that $\mathrm{vol}(M_p)/\mathrm{vol}(N_p) = p$. \square

Remark. Note that it is immediate the manifolds constructed in Theorem 1.1 are non-isometric since their volumes are different. Also note that the choice of metric on M is irrelevant.

Theorem 1.2 will follow immediately from the following slightly stronger result that gets more than pairs of length equivalent manifolds. In particular this completes the proof of Theorem 1.1 for the case of hyperbolic surfaces (closed or otherwise).

Theorem 4.1. *Let M be a Riemannian manifold whose fundamental group is large. For every integer $n \geq 2$ and every odd prime p , there exists a finite tower of covers*

$$M_n \longrightarrow M_{n-1} \longrightarrow \dots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M$$

such that $L(M_j) = L(M_k)$ for $1 \leq j, k \leq n$ and $M_j \longrightarrow M_{j-1}$ is degree p .

Proof. Let Γ be a finite index subgroup of $\pi_1(M)$ that surjects F_2 , the non-abelian free group of rank 2. Now for all integers n , the groups $\mathrm{SL}(n+1; \mathbb{F}_p)$ are 2-generator by [16]. Hence Γ surjects all such groups. By the final part of the argument in the proof of Theorem 1.1 it now follows that, by passage to a further subgroup of finite index, we can arrange a finite cover X of M with

$$\pi_1(X) \twoheadrightarrow \mathbb{F}_p^n \rtimes \mathrm{SL}(n; \mathbb{F}_p).$$

For this we use the injection $\mathbb{F}_p^n \rtimes \mathrm{SL}(n; \mathbb{F}_p) \rightarrow \mathrm{SL}(n+1; \mathbb{F}_p)$ defined by

$$(v, A) \mapsto \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}.$$

Consider any complete \mathbb{F}_p -flag

$$0 \subset V_1 \subset V_2 \subset \dots \subset V_{n-1} \subset V_n = \mathbb{F}_p^n$$

in \mathbb{F}_p^n . By Lemma 3.2, for $j \neq k$, (V_j, V_k) is a conjugacy covering pair in $\mathbb{F}_p^n \rtimes \mathrm{SL}(n; \mathbb{F}_p)$. Hence Theorem 2.1 produces covers

$$M_n \rightarrow M_{n-1} \rightarrow \dots \rightarrow M_2 \rightarrow M_1 \rightarrow X \rightarrow M$$

such that $L(M_j) = L(M_{j-1})$ and the degree of the covering $M_j \rightarrow M_{j-1}$ is p . \square

5 Final Remarks

We conclude by pointing out some other consequences of our methods.

1 We shall denote by \mathbf{H}_Y^n the n -dimensional hyperbolic spaces modelled on $Y \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ (where $n = 2$ when $Y = \mathbb{O}$).

Theorem 5.1. *Let Γ be a torsion-free lattice in $\mathrm{Isom}(\mathbf{H}_Y^n)$. Then there exist infinitely many pairs of finite covers of $\{M_j, N_j\}$ of $M = \mathbf{H}_Y^n/\Gamma$ such that $L(M_j) = L(N_j)$ and for which $\mathrm{vol}(M_j)/\mathrm{vol}(N_j) \rightarrow \infty$.*

Proof. As in the proof of Theorem 1.1 an application of [18], [9] in combination with the Chebotarev Density Theorem provides infinitely primes p such that Γ surjects onto certain finite groups $\mathbf{G}(R)$ of Lie type. That $\mathbf{G}(R)$ contains a subgroup isomorphic to $\mathrm{PSL}(2; R)$ follows from an argument analogous to that given in the proof of Theorem 1.1; namely the inclusion of $\mathrm{PO}_0(2, 1; S)$ into $\mathrm{PO}_0(m, 1; S)$ where S is a finite extension of \mathbb{Z} , together with the isomorphism between $\Omega(2, 1; R)$ and $\mathrm{PSL}(2; R)$ established in the proof of Theorem 1.1. Hence, for all cases of Theorem 5.1, these remarks provide subgroups as in Example 2, and the argument proceeds as in the proof of Theorem 1.1. \square

As is clear from this discussion (and the generality of [18] and [9]) our methods also apply to lattices in every non-compact higher rank simple Lie group. By [8] Proposition 4 Window 2 we can arrange for the finite groups of Lie type occurring in Strong Approximation to contain a group as in Example 1. In particular [8] Proposition 4 Window 2 yields a copy of $(\mathbf{P})\mathrm{SL}(3; \mathbb{F}_p)$.

2 Using reduction homomorphisms arising from products of prime ideals and the methods implemented in the proof of Theorem 1.1, we can obtain large sequences of length equivalent hyperbolic manifolds.

Theorem 5.2. *For every finite volume hyperbolic m -manifold M , every $k > 0$, and every $n > 0$, there exists a tower of finite covers*

$$M_n \longrightarrow M_{n-1} \longrightarrow \dots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M$$

such that $L(M_i) = L(M_j)$ for $1 \leq i, j \leq n$ and $\mathrm{vol}(M_{j+1})/\mathrm{vol}(M_j) > k$.

This too can be extended to manifolds modelled on non-compact simple Lie groups with finite center.

3 Theorem 4.1 includes many examples. For instance this applies to surfaces with negative Euler characteristic equipped with any Riemannian metric and hyperbolizable 3-manifolds with at least one boundary component of genus at least two (these are large by [2]). Also, any large finitely presented group is the fundamental group of a Riemannian n -manifold for any $n \geq 4$, and so Theorem 4.1 applies to a large class of Riemannian n -manifolds.

4 The methods also apply to produce hyperbolic m -manifolds which have the same *weak complex length spectrum*. Recall that the *complex length* of a closed geodesic γ in a hyperbolic m -manifold is a pair $(\ell(\gamma), V)$ where $\ell(g)$ is the length of γ and $V \in \mathrm{O}(m-1)$ is determined by the holonomy of γ . The complex length spectrum is the collection of such complex lengths with multiplicities and the weak complex length spectrum forgets multiplicities as before. The proof of Theorem 2.1 applies to give the manifolds with the same weak complex length spectrum (see [13] for more on the complex length spectrum).

5 We finish with a question. The known methods of producing iso-length spectral hyperbolic manifolds result in commensurable manifolds and it is an open question as to whether this is always the case. By construction, the length equivalent hyperbolic manifolds constructed here are also commensurable.

Question: Let M_1 and M_2 be length equivalent hyperbolic manifolds of finite volume. Are they commensurable?

It is shown in [1] that if the manifolds M_1 and M_2 are arithmetic hyperbolic 3-manifolds, then this question has an affirmative answer.

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Department of Mathematics,
University of Illinois at Urbana-Champaign,
Urbana, IL 61801.
email: clein@math.uiuc.edu

Department of Mathematics,
University of Texas
Austin, TX 78712.
email: dmcreyn@math.utexas.edu, areid@math.utexas.edu