Some Extremal Functions in Fourier Analysis

Emanuel Carneiro and Jeffrey D. Vaaler

Department of Mathematics
University of Texas at Austin

24th Southeastern Analysis Meeting
Vanderbilt University - Nashville, TN - March, 2008

The Extremal Problem

Given a function $f : \mathbb{R} \to \mathbb{R}$, we seek an entire function $K : \mathbb{C} \to \mathbb{C}$ of exponential type $2\pi\delta$, ($|K(z)| \leq A_\epsilon e^{(2\pi\delta+\epsilon)|z|}$) such that the integral

$$\int_{-\infty}^{\infty} |K(x) - f(x)| \, \mathrm{d}x$$

is minimal. We may impose the following conditions:

- Majorizing Problem $K(x) \geq f(x)$, for all $x \in \mathbb{R}$.
- Minorizing Problem $K(x) \leq f(x)$, for all $x \in \mathbb{R}$.
- Best Approximation Problem No restrictions.

Solution of the Extremal Problem

By a solution of this problem we mean:

- Prove the existence and uniqueness of $K(z)$, and if possible, find an explicit expression.
- Find the minimal value of the integral

$$\int_{-\infty}^{\infty} |K(x) - f(x)| \,\mathrm{d}x.$$

- Define

$$\Phi(x) = K(x) - f(x).$$

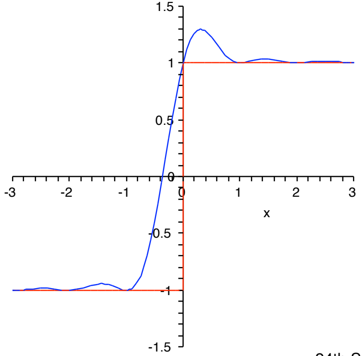
  Describe the Fourier transform $\widehat{\Phi}(t)$.

Example I - A. Beurling (1930's)

- The function

$$B(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n\in\mathbb{Z}} \frac{\mathrm{sgn}_+(n)}{(z-n)^2} + \frac{2}{z} \right\}$$

is the extremal majorant of exponential type 2π of $f(x) = \mathrm{sgn}(x)$, with $\int_{-\infty}^{\infty} \{B(x) - \mathrm{sgn}(x)\}\, \mathrm{d}x = 1$.

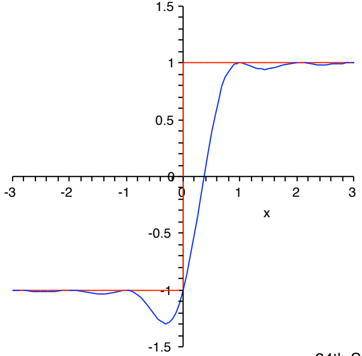


Example II - A. Beurling(1930's)

- The function

$$B_1(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n\in\mathbb{Z}} \frac{\mathrm{sgn\_}(n)}{(z-n)^2} + \frac{2}{z} \right\}$$

is the extremal minorant of exponential type 2π of $f(x) = sgn(x)$, with $\int_{-\infty}^{\infty} \{\mathrm{sgn}(x) - B_1(x)\}\, \mathrm{d}x = 1$.

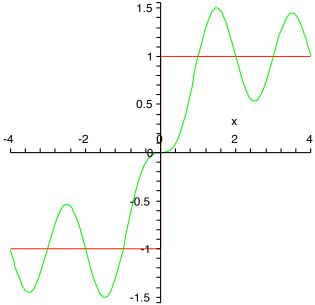


Example III - A. Beurling (1930's)

- The function

$$B_2(z) = \left(\frac{\sin \pi z}{\pi}\right) \left\{ \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{(-1)^n \operatorname{sgn}(n) z}{(z-n)n} + \log 4 \right\}$$

is the best approximation of exponential type π of $f(x) = sgn(x)$, with $\int_{-\infty}^{\infty} |B_2(x) - \operatorname{sgn}(x)| \, \mathrm{d}x = 1$.



Motivation

Why would one study such problem?

- The majorizing property together with the fact that the Fourier transform of $K(x)$ is "morally" supported in $[-\delta, \delta]$ (Paley-Wiener theorem) lead to inequalities in analysis, analytic number theory and uniform distribution.
- For example, suppose $K(x) \geq f(x)$. Define

$$\Phi(x) = K(x) - f(x).$$

  We should have

$$\widehat{\Phi}(t) = -\widehat{f}(t), \quad \text{for} \quad |t| \geq \delta.$$

Hilbert's Inequality I

- Let $\{\lambda_n\}_{n=1}^N$ be well-spaced real numbers, $|\lambda_n - \lambda_m| \geq \delta$.
- Let $\{a_n\}_{n=1}^N$ be complex numbers. Then (here $e(x) = e^{2\pi i x}$)

$$\begin{aligned}
0 &\leq \int_{-\infty}^{\infty} \Phi(x) \left| \sum_{n=1}^{N} a_n e(\lambda_n x) \right|^2 \\
&= \int_{-\infty}^{\infty} \Phi(x) \sum_{m,n} a_n \overline{a_m} e((\lambda_n - \lambda_m)x) \\
&= \sum_{m,n} a_n \overline{a_m} \int_{-\infty}^{\infty} \Phi(x) e((\lambda_n - \lambda_m)x) \\
&= \sum_{m,n} a_n \overline{a_m} \widehat{\Phi}(\lambda_m - \lambda_n) \\
&= \widehat{\Phi}(0) \sum_{n=1}^{N} |a_n|^2 - \sum_{m \neq n}^{N} a_n \overline{a_m} \widehat{f}(\lambda_m - \lambda_n)
\end{aligned}$$

Hilbert's Inequality II

- We therefore obtain (recall $\Phi(x) = K(x) - f(x) \geq 0$)

$$\sum_{m \neq n}^{N} a_n \overline{a_m} \widehat{f}(\lambda_m - \lambda_n) \leq \widehat{\Phi}(0) \sum_{n=1}^{N} |a_n|^2.$$

- In the case $f(x) = \mathrm{sgn}(x)$ we have $\widehat{f}(t) = \frac{1}{\pi i t}$ (Hilbert transform sense) and we obtain, for example,

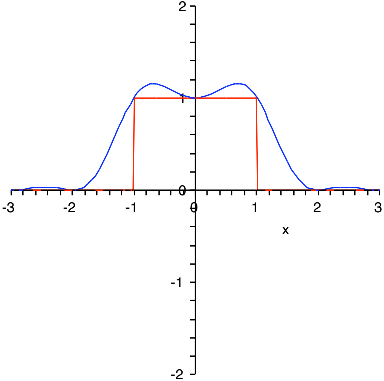
$$\left| \sum_{\substack{m,n=1 \\ m \neq n}}^{N} \frac{a_n \overline{a_m}}{m - n} \right| \leq \pi \sum_{n=1}^{N} |a_n|^2.$$

This is known as Hilbert's inequality (Hilbert 1900's, Schur 1906, Montgomery and Vaughan 1974).

Example IV - A. Selberg

- Later, A. Selberg recognized that Beurling's functions could be used to majorize the characteristic function of an interval:

$$\chi_{[a,b]}(x) = \frac{1}{2}\{\mathrm{sgn}(x-a) + \mathrm{sgn}(b-x)\}$$
$$C(x) = \frac{1}{2}\{B(x-a) + B(b-x)\}.$$



Large Sieve Inequality - A. Selberg

- Selberg used this fact to give a simple proof of the Large Sieve inequality in analytic number theory. Let

$$S(x) = \sum_{n=M+1}^{M+N} a_n e(nx)$$

be a trigonometric polynomial with period 1, and let $\xi_1, \xi_2, ..., \xi_R$ be real numbers well-spaced modulo 1, i.e., $\|\xi_r - \xi_s\| \geq \delta > 0$ for $r \neq s$. Then

$$\left| \sum_{r=1}^{R} S(\xi_r) \right|^2 \leq (N - 1 + \delta^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2$$

The function $e^{-\lambda|x|}$

In 1981, S. Graham and J.D. Vaaler solved the majorizing and minorizing problem for the even function $f(x) = e^{-\lambda|x|}$ where $\lambda > 0$ is a positive parameter.

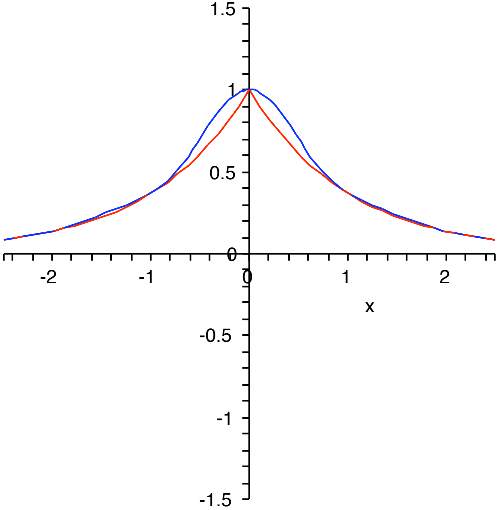
- Majorant interpolates at the integers:

$$M(\lambda, z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{k\in\mathbb{Z}} \frac{e^{-\lambda|k|}}{(z-k)^2} - \lambda \sum_{l\in\mathbb{Z}} \frac{\mathrm{sgn}(l) e^{-\lambda|l|}}{(z-l)} \right\}.$$

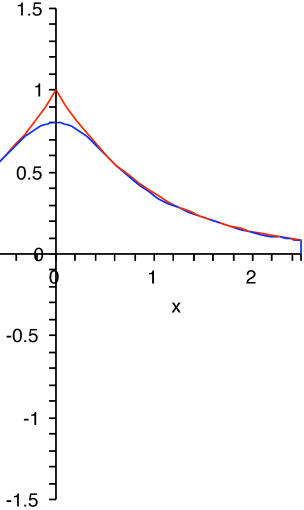
- Minorant interpolates at the integers plus a half:

$$L(\lambda, z) = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \sum_{k\in\mathbb{Z}} \frac{e^{-\lambda|k+\frac{1}{2}|}}{(z-k-\frac{1}{2})^2} - \lambda \sum_{l\in\mathbb{Z}} \frac{\mathrm{sgn}(l+\frac{1}{2}) e^{-\lambda|l+\frac{1}{2}|}}{(z-l-\frac{1}{2})} \right\}.$$

Majorant of $e^{-\lambda|x|}$



Minorant of $e^{-\lambda|x|}$



Integrating on the parameter λ

- We have $L(\lambda, x) \leq e^{-\lambda|x|} \leq M(\lambda, x)$ with

$$\int_{-\infty}^{\infty} \left\{ e^{-\lambda|x|} - L(\lambda, x) \right\} \mathrm{d}x = \frac{2}{\lambda} - \operatorname{csch}\left(\frac{\lambda}{2}\right),$$

$$\int_{-\infty}^{\infty} \left\{ M(\lambda, x) - e^{-\lambda|x|} \right\} \mathrm{d}x = \coth\left(\frac{\lambda}{2}\right) - \frac{2}{\lambda}.$$

- Let μ be a measure defined on the Borel subsets of $(0, \infty)$. One might expect to show that

$$L_\mu(z) = \int_0^\infty L(\lambda, z)\, \mathrm{d}\mu(\lambda) \quad \textit{and} \quad M_\mu(z) = \int_0^\infty M(\lambda, z)\, \mathrm{d}\mu(\lambda)$$

solve the extremal problem for $f_\mu(x) = \int_0^\infty e^{-\lambda|x|} \mathrm{d}\mu(\lambda)$

Integrating on the parameter λ

- We have $L(\lambda, x) \leq e^{-\lambda|x|} \leq M(\lambda, x)$ with

$$\begin{aligned}\int_{-\infty}^{\infty} \left\{ e^{-\lambda|x|} - L(\lambda, x) \right\} \mathrm{d}x &= \frac{2}{\lambda} - \operatorname{csch}\left(\frac{\lambda}{2}\right), \\ \int_{-\infty}^{\infty} \left\{ M(\lambda, x) - e^{-\lambda|x|} \right\} \mathrm{d}x &= \coth\left(\frac{\lambda}{2}\right) - \frac{2}{\lambda}.\end{aligned}$$

- Let μ be a measure defined on the Borel subsets of $(0, \infty)$. One might expect to show that

$$L_\mu(z) = \int_0^\infty L(\lambda, z)\,\mathrm{d}\mu(\lambda) \quad \textit{and} \quad M_\mu(z) = \int_0^\infty M(\lambda, z)\,\mathrm{d}\mu(\lambda)$$

solve the extremal problem for $f_\mu(x) = \int_0^\infty e^{-\lambda|x|}\mathrm{d}\mu(\lambda)$

Integrating on the parameter λ

Theorem (Graham-Vaaler,1981)

Let μ be a finite measure such that

$$\int_0^\infty \lambda^{-1} \mathrm{d}\mu(\lambda) < \infty$$

Then $M_\mu(z)$ and $L_\mu(z)$ are the unique extremals of exp. type 2π of $f_\mu(x)$. Moreover

$$\begin{aligned} \int_{-\infty}^{\infty} \{f_\mu(x) - L_\mu(x)\}\, \mathrm{d}x &= \int_0^\infty \left\{\tfrac{2}{\lambda} - \operatorname{csch}\left(\tfrac{\lambda}{2}\right)\right\} \mathrm{d}\mu(\lambda), \\ \int_{-\infty}^{\infty} \{M_\mu(x) - f_\mu(x)\}\, \mathrm{d}x &= \int_0^\infty \left\{\coth\left(\tfrac{\lambda}{2}\right) - \tfrac{2}{\lambda}\right\} \mathrm{d}\mu(\lambda). \end{aligned}$$

New Idea

- We can "subtract a constant from both sides" to consider a wider class of measures. For this define $\widetilde{f}_\mu : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, by

$$\widetilde{f}_\mu(x) = \int_0^\infty \{e^{-\lambda|x|} - e^{-\lambda}\} \mathrm{d}\mu(\lambda).$$

- The extremal candidates are now defined in terms of $\widetilde{f}_\mu$:

$$\widetilde{L}_\mu(z) = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\widetilde{f}_\mu(n-\frac{1}{2})}{(z-n+\frac{1}{2})^2} + \sum_{n=-\infty}^{\infty} \frac{\widetilde{f}'_\mu(n-\frac{1}{2})}{(z-n+\frac{1}{2})} \right\},$$

$$\widetilde{M}_\mu(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n\in\mathbb{Z}} \frac{\widetilde{f}_\mu(n)}{(z-n)^2} + \sum_{n\neq 0} \frac{\widetilde{f}'_\mu(n)}{(z-n)} \right\}.$$

New Idea

- We can "subtract a constant from both sides" to consider a wider class of measures. For this define $\widetilde{f}_\mu : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, by

$$\widetilde{f}_\mu(x) = \int_0^\infty \{e^{-\lambda|x|} - e^{-\lambda}\} \mathrm{d}\mu(\lambda).$$

- The extremal candidates are now defined in terms of $\widetilde{f}_\mu$:

$$\widetilde{L}_\mu(z) = \left(\frac{\cos \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\widetilde{f}_\mu(n-\frac{1}{2})}{(z-n+\frac{1}{2})^2} + \sum_{n=-\infty}^{\infty} \frac{\widetilde{f}'_\mu(n-\frac{1}{2})}{(z-n+\frac{1}{2})} \right\},$$

$$\widetilde{M}_\mu(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n\in\mathbb{Z}} \frac{\widetilde{f}_\mu(n)}{(z-n)^2} + \sum_{n\neq 0} \frac{\widetilde{f}'_\mu(n)}{(z-n)} \right\}.$$

Main Result

Theorem

- *If the measure* μ *satisfies* $\int_0^\infty \frac{\lambda}{\lambda^2+1} \mathrm{d}\mu(\lambda) < \infty$ *then* $\widetilde{L}_\mu(z)$ *is the unique extremal minorant of exp. type* 2π *for* $\widetilde{f}_\mu(x)$*, and satisfies*

$$\int_{-\infty}^{\infty} \left\{ \widetilde{f}_\mu(x) - \widetilde{L}_\mu(x) \right\} \mathrm{d}x = \int_0^\infty \left\{ \tfrac{2}{\lambda} - \operatorname{csch}\left(\tfrac{\lambda}{2}\right) \right\} \mathrm{d}\mu(\lambda).$$

- *If the measure* μ *satisfies* $\int_0^\infty \frac{\lambda}{\lambda+1} \mathrm{d}\mu(\lambda) < \infty$ *then* $\widetilde{M}_\mu(z)$ *is the unique extremal majorant of exp. type* 2π *for* $\widetilde{f}_\mu(x)$*, and satisfies*

$$\int_{-\infty}^{\infty} \left\{ \widetilde{M}_\mu(x) - \widetilde{f}_\mu(x) \right\} \mathrm{d}x = \int_0^\infty \left\{ \coth\left(\tfrac{\lambda}{2}\right) - \tfrac{2}{\lambda} \right\} \mathrm{d}\mu(\lambda).$$

Special Cases

We highlight here two interesting applications of this result:

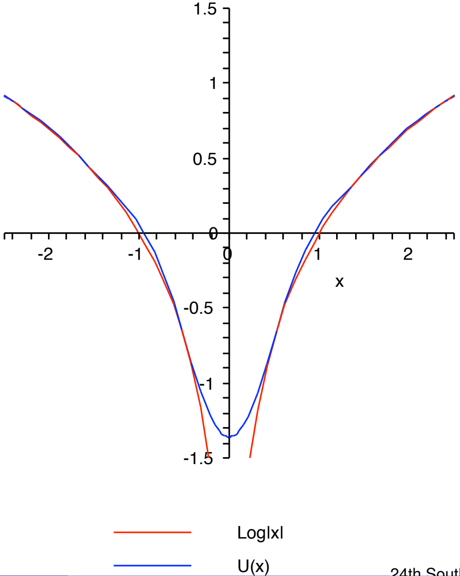
- If we consider $\mathrm{d}\mu(\lambda) = \lambda^{-1}\mathrm{d}\lambda$ we obtain:

$$\widetilde{f}_{\mu}(x) = -\log|x|$$

- If we consider $\mathrm{d}\mu(\lambda) = \lambda^{-\sigma}\mathrm{d}\lambda$, with $0 < \sigma < 2$ and $\sigma \neq 1$ we obtain:

$$\widetilde{f}_{\mu}(x) = \Gamma(1-\sigma)\{|x|^{\sigma-1} - 1\}$$

Majorant of logarithm



Hilbert-type Inequalities

Corollary

Let $\xi_1, \xi_2, \ldots, \xi_N$ be real numbers such that $0 < \delta \leq |\xi_m - \xi_n|$ whenever $m \neq n$. Let $a_1, a_2, \ldots, a_N$ be complex numbers. If $0 < \sigma < 1$ then

$$-\frac{(2 - 2^{2-\sigma})\zeta(\sigma)}{\delta^\sigma} \sum_{n=1}^{N} |a_n|^2 \leq \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \bar{a}_n}{|\xi_m - \xi_n|^\sigma} \leq HLS \qquad (1)$$

if $\sigma = 1$ then

$$-\frac{\log 4}{\delta} \sum_{n=1}^{N} |a_n|^2 \leq \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \bar{a}_n}{|\xi_m - \xi_n|}, \qquad (2)$$

Corollary (cont.)

If $1 < \sigma < 2$ *then*

$$-\frac{(2-2^{2-\sigma})\zeta(\sigma)}{\delta^{\sigma}}\sum_{n=1}^{N}|a_n|^2 \leq \sum_{\substack{m,n=1\\ n\neq m}}^{N}\frac{a_m\overline{a}_n}{|\xi_m-\xi_n|^{\sigma}} \leq \frac{2\zeta(\sigma)}{\delta^{\sigma}}\sum_{n=1}^{N}|a_n|^2, \quad (3)$$

where ζ *denotes the Riemann zeta-function. All the constants appearing are sharp.*

Erdös-Turán Inequality

Corollary

Let $F_M(z)$ be the monic polynomial defined by

$$F_M(z) = \prod_{m=1}^{M} (z - \alpha_m),$$

and assume that $|\alpha_m| \leq 1$ for each $m = 1, 2, \ldots, M$. Then for each nonnegative integer N we have

$$\sup_{|z|\leq 1} \log |F_M(z)| \leq M(N+1)^{-1} \log 2 + \sum_{n=1}^{N} n^{-1} \left| \sum_{m=1}^{M} (\alpha_m)^n \right|.$$

papers available at www.math.utexas.edu/users/ecarneiro

Thank you!

E. Carneiro and J. D. Vaaler,
Some extremal functions in Fourier analysis, II,
submitted.

S. W. Graham and J. D. Vaaler,
A class of Extremal Functions for the Fourier Transform,
Tran. Amer. Math. Soc. 265, (1981), 283–382.

F. Littmann,
Entire majorants via Euler-Maclaurin summation,
Trans. Amer. Math. Soc. 358, (2006), no. 7, 2821–2836.

H. L. Montgomery and R. C. Vaughan,
Hilbert's Inequality,
J. London Math. Soc. (2), 8, (1974), 73–81.

J. D. Vaaler,
Some Extremal Functions in Fourier Analysis,
Bull. Amer. Math. Soc. 12, (1985), 183–215.