On the regularity of maximal operators

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Outline

The Hardy-Littlewood maximal operator

Definition

Let $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{R>0} ⨍_{B(x,R)} |f(y)| dy$$

where $B(x,R)$ denotes the ball of radius R centered in x. For $\Omega \subset \mathbb{R}^n$ a proper open subset of $\mathbb{R}^n$ and $f \in L^1_{loc}(\Omega)$ we can define the local maximal operator M_Ω at a point $x \in \Omega$ by

$$M_\Omega f(x) = \sup_{0<R<\mathrm{dist}(x,\partial\Omega)} ⨍_{B(x,R)} |f(y)| dy$$

The Hardy-Littlewood maximal operator

- The classical results are that $M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and $M_\Omega : L^p(\Omega) \to L^p(\Omega)$ are bounded for $p > 1$, namely

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \quad \textit{and} \quad \|M_\Omega f\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$

- In this case we also have sublinearity

$$M(f+g)(x) \leq Mf(x) + Mg(x)$$

- Boundedness + Sublinearity $\Rightarrow$ Continuity in L^p

$$\|Mf_j - Mf\|_{L^p} \leq \|M(f_j - f)\|_{L^p} \leq C\|f_j - f\|_{L^p}$$

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Maximal operator on Sobolev spaces

- Question: What happens when the maximal operator is applied to a function that has a weak derivative?

Theorem (Kinnunen,1997)

Let $f \in W^{1,p}(\mathbb{R}^n)$, $p > 1$. Then Mf has a weak derivative ∇Mf and it satisfies

$$|\nabla Mf(x)| \leq M(|\nabla f|)(x) \quad a.e \ \ in \ \ \mathbb{R}^n$$

This proves that $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is bounded

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Maximal operator on Sobolev spaces: local version

Theorem (Kinnunen and Lindqvist,1998)

Let $f \in W^{1,p}(\Omega)$, $p > 1$. Then $M_\Omega f$ has a weak derivative $\nabla M_\Omega f$ and it satisfies

$$|\nabla M_\Omega f(x)| \leq 2M_\Omega(|\nabla f|)(x) \quad a.e \;\; in \;\; \Omega$$

This proves that $M_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is bounded

$$\|M_\Omega f\|_{W^{1,p}(\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)}$$

Remark: M_Ω preserves boundary values.

Proofs of these results:

- Functional analysis arguments.
- Uses the reflexivity of L^p and $W^{1,p}$.

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Sketch of the proof

- Suppose $f \geq 0$, since $|D_i|f|| = |D_i f|$ a.e. $\quad i = 1, 2, ..., n.$
- Enumerate the rationals: $r_1, r_2, r_3, ...$ and consider

$$\varphi_k(x) = ⨍_{B(x,r_k)} f(y)dy$$

$$D_i\varphi_k(x) = ⨍_{B(x,r_k)} D_i f(y)dy \Rightarrow |D_i\varphi_k(x)| \leq MD_i f(x) \quad a.e.$$

- Now let

$$g_k(x) = \sup_{1\leq j\leq k} \varphi_j(x) \Rightarrow |D_i g_k(x)| \leq MD_i f(x) \quad a.e. \qquad (1)$$

- $|g_k|_{W^{1,p}} \leq C \Rightarrow$ exists subsequence $g_k \rightharpoonup g \in W^{1,p}$. But $g_k(x) \to Mf(x)$ pointwise $\Rightarrow$ $g = Mf$
- From (1) we conclude that $|D_i Mf(x)| \leq MD_i f(x)$. □

Continuity in $W^{1,p}$

- Continuity does not follow directly from boundedness because of the lack of sublinearity (on the derivatives).

Theorem (Luiro, 2007)

$M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is a continuous operator.

- Open question 1: For $\Omega \subset \mathbb{R}^n$ open and proper, is $M : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ continuous?

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The bilinear maximal operator

Definition

For $\alpha \neq 1$ define the bilinear maximal operator

$$\mathcal{M}(f,g)(x) = \sup_{R>0} \fint_{B_R} |f(x-\alpha y)g(x-y)|dy$$

- Holder's inequality $\Rightarrow \mathcal{M} : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ if $1/p + 1/q = 1/r$ and $r > 1$.

Theorem (M. Lacey, 2000)

$\mathcal{M}$ maps $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ if $1 < p, q < \infty$, $1/p + 1/q = 1/r$ and $r > 2/3$.

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Bilinear maximal on Sobolev spaces

On the spirit of Kinnunen's work we may ask ourselves: what happens when the bilinear maximal operator is applied to Sobolev functions?

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Given $\alpha \neq 1$, the bilinear maximal operator $\mathcal{M}$ maps $W^{1,p}(\mathbb{R}^n) \times W^{1,q}(\mathbb{R}^n) \to W^{1,r}(\mathbb{R}^n)$ boundedly and continuously, where $1/p + 1/q = 1/r$, $1 < p, q < \infty$ and

(a) $r \geq 1$, *if* $n = 1$;

(b) $r > 1$, *if* $n > 1$.

Boundedness is a consequence of the following pointwise estimate

$$|\nabla \mathcal{M}(f,g)(x)| \leq \mathcal{M}(f, |\nabla g|)(x) + \mathcal{M}(|\nabla f|, g)(x) \quad a.e.\ x \in \mathbb{R}^n$$

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Comments on the proof

- New feature: Results for L^1. No more functional analysis tools.
- Proof holds if Lacey's result is valid for $n > 1$.
- Main ideas:
  - Boundedness: Prove absolute continuity over lines (inspired by Hajlaz and Onninen, 2004)

$$|\mathcal{M}(f,g)(x) - \mathcal{M}(f,g)(y)| \leq \int_{\overline{xy}} \mathcal{M}(f,|\nabla g|) + \mathcal{M}(|\nabla f|,g)$$

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No almost everywhere continuity

Example (1)

The maximal operators $M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and $M_\Omega : L^p(\Omega) \to L^p(\Omega)$ for $p > 1$ do not preserve pointwise convergence almost everywhere.

For instance:

$$u_k(x) = \frac{1}{m(B_{\frac{1}{k}})} \chi_{B_{\frac{1}{k}}}(x)$$

- Issues about the stability of the weak convergence under nonlinear operators are much more interesting and have been studied in [Moreira and Teixeira, 2005] for a certain class called Nemytskii nonlinearites, with applications to differential equations in [Teixeira, 2005] and [Caffarelli, Salsa and Silvestre, 2008].

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No weak continuity in L^p

The question is: if $u_k \rightharpoonup u$ do we have $T(u_k) \rightharpoonup T(u)$?

Example (2)

The maximal operators $M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and $M_\Omega : L^p(\Omega) \to L^p(\Omega)$ for $p > 1$ are not sequentially weakly continuous.

- local case: $\Omega = (-1, 1) \subset \mathbb{R}$. Take $u_n(x) = \sin(2\pi nx)$ in $L^2(\Omega)$.
- global case: In $L^2(\mathbb{R})$ take

$$u_n(x) = \frac{\sin(2\pi nx)}{1 + x^2}$$

Weak continuity in $W^{1,p}(\Omega)$

Proposition (3)

Suppose Ω is a bounded domain with Lipschitz boundary. Then, the local maximal operator $\mathcal{M}_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is sequentially weakly continuous for $p > 1$.

Proof: Let $f_j \rightharpoonup f$ in $W^{1,p}(\Omega)$. The sequence $\mathcal{M}_\Omega(f_j)$ must admit a weakly convergent subsequence, by reflexivity. Assume

$$\mathcal{M}_\Omega(f_j) \rightharpoonup g \ \text{ in } \ W^{1,p}(\Omega)$$

By the compactness of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and the continuity in $L^p(\Omega)$ of the local maximal operator, we have

$$\mathcal{M}_\Omega(f_j) \to \mathcal{M}_\Omega(f) \ \text{ in } \ L^p(\Omega)$$

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Weak continuity in $W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$

Theorem (C. - Moreira, 2008)

Let $1 < p < \infty$ and suppose $u_k \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$. There exists a subsequence $M(u_{k_j}) \to M(u)$ a.e. in $\mathbb{R}^n$.

Corollary (4)

Assume $1 < p < \infty$. The maximal operator $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is sequentially weakly continuous.

Proof: Let $u_k \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$. By the boundeness of the maximal operator, we can assume $M(u_k) \rightharpoonup g$ in $W^{1,p}(\mathbb{R}^n)$. By the previous Theorem, there exists a subsequence $M(u_{k_j}) \to M(u)$ a.e in $\mathbb{R}^n$. This is sufficient to conclude that $M(u) = g$. □

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Sketch of the proof

- By the sublinearity, it is enough to prove the case $u \equiv 0$.
- Let us consider $B = B_L(0)$, where $L > 0$.
  For each $m = 1, 2, 3, ...$ we can take $R_m > 0$ large enough so that, for every $x \in \mathbb{R}^n$

$$\fint_{B_R(x)} |u_k(y)| dy \leq \frac{1}{m} \quad \textit{for all} \quad k \in \mathbb{N} \quad \textit{whenever} \quad R \geq R_m.$$

- Consider $B_m^\star := B_{L+2R_m}(0)$ and the local maximal operator with respect to this ball. For any $x \in B$, if $\delta_x = \mathrm{dist}(x, \partial B_m^\star)$:

$$\begin{aligned} M(u_k)(x) &= \max\left\{ \sup_{0<R<\delta_x} \fint_{B_R(x)} |u_k(y)| dy, \sup_{R>\delta_x} \fint_{B_R(x)} |u_k(y)| dy \right\} \\ &\leq \max\left\{ M_{B_m^\star}(u_k)(x), \frac{1}{m} \right\} \quad (*) \end{aligned}$$

Sketch of the proof- cont.

- Since $W^{1,p}(B_m^\star) \hookrightarrow L^p(B_m^\star)$ compactly, $M_{B_m^\star}(u_k) \to 0$ in $L^p(B_m^\star)$. Therefore, there is a subsequence $M_{B_m^\star}(u_{k_j}^m) \to 0$ *a.e.* in $B_m^\star$. From (*) we conclude that

$$\limsup_{j\to\infty} M(u_{k_j}^m)(x) \leq \frac{1}{m} \quad \text{a.e. in } B.$$

- Using the Cantor diagonal argument we can find a subsequence $\{u_{k_j}\}$ such that

$$M(u_{k_j})(x) \to 0 \quad \text{a.e. in } B = B_L(0).$$

- Since the original ball B was arbitrary, we can use once more the Cantor diagonal argument applied to $\mathbb{R}^n = \bigcup_{n=0}^{\infty} B_n(0)$ to conclude the Theorem. $\square$

Summary table - Weak continuity

- Arrows indicate that the Hardy-Littlewood maximal operator is bounded.
- Red arrows = not seq. weakly continuous.
- Black arrows: seq. weakly continuous.

$$\begin{array}{ccc} L^p(\Omega) & \rightarrow & L^p(\Omega) \\ & \nearrow & \\ W^{1,p}(\Omega) & \rightarrow & W^{1,p}(\Omega) \end{array} \qquad \begin{array}{ccc} L^p(\mathbb{R}^n) & \rightarrow & L^p(\mathbb{R}^n) \\ & \nearrow & \\ W^{1,p}(\mathbb{R}^n) & \rightarrow & W^{1,p}(\mathbb{R}^n) \end{array}$$

Open problems

(1) Is $M : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ continuous?

(2) Is M. Lacey's result valid for dimension $n > 1$?

(3) Conjecture: Let $f \in W^{1,1}(\mathbb{R}^n)$, then Mf has a weak derivative and $\|DMf\|_{L^1} \leq C\|Df\|_{L^1}$?

- Proved true for the non-centered maximal operator in dimension $n = 1$ [Tanaka, 2002].
- J. Aldaz and J. Perez Lazaro in 2007 proved that if $f : \mathbb{R} \to \mathbb{R}$ is BV then the non-centered maximal Mf is absolutely continuous and

$$\|DMf\|_{L^1(\mathbb{R})} \leq |Df|(\mathbb{R})$$

Open problems - cont.

(4) About the inequality $\|Mf\|_{W^{1,p}(\mathbb{R}^n)} \leq C_p \|f\|_{W^{1,p}(\mathbb{R}^n)}$
- Does it have extremals?
- Can we find sharp constants? (maybe $n = 1$, with the non-centered maximal?)

(5) Study regularity properties of other maximal and singular integral operators.

(6) Applications of these results to PDE's?

(7) What happens when we change the underlying space for a generic Riemannian manifold?

Thank you!

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