

A Sharp Strichartz Inequality

Emanuel Carneiro

Department of Mathematics
University of Texas at Austin

Analysis Seminar
University of Texas at Austin - October 2008

Sharp Inequalities

- Hausdorff-Young inequality for the Fourier transform (Beckner '75)
- Young's inequality for convolutions (Beckner '75)
- Hardy-Littlewood-Sobolev inequality (Lieb '83)
- Sobolev inequalities (Aubin, Talenti, Beckner and others)

Program - Sharp Strichartz

Let $u : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}$ be the solution of the Schrödinger equation

$$\begin{cases} iu_t + \Delta u &= 0 \\ u(0, x) &= f(x). \end{cases} \quad (1)$$

Strichartz inequalities:

$$\|u(t, x)\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)},$$

for (q, r) satisfying $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$, $2 \leq q, r \leq \infty$ and $(q, r, n) \neq (2, \infty, 2)$.

Recall:

$$\|u(t, x)\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} = \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t, x)|^r dx \right)^{q/r} dt \right]^{1/q}.$$

Main Goal: Find the constant $C(q, r) = \sup_{f \neq 0} \frac{\|u(t, x)\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}}$.

History

- (Kunze '03) A maximizer exists in the case $n = 1, (q, r) = (6, 6)$.
- (Foschi and Hundertmark-Zharnitsky '06) Calculated the constants in the cases $n = 1, (q, r) = (6, 6)$ and $n = 2, (q, r) = (4, 4)$, showing that the only maximizers are Gaussians.

Conjecture: In the case $n \geq 3, q = r = 2 + 4/n$, the only maximizers are Gaussians and the sharp constant is

$$C = \left(\frac{1}{2} \left(\frac{n}{n+2} \right)^{n/2} \right)^{n/(2n+4)}.$$

- (Shao '08) Maximizers **do exist** for all non-endpoint Strichartz estimates ($q \neq 2$ if $n \geq 3$ and $q \neq 4$ if $n = 1$).

Main Result

Theorem

For $k \in \mathbb{Z}$, $k \geq 2$ and $(n, k) \neq (1, 2)$ we have

$$\|u(t, \mathbf{x})\|_{L_t^{2k} L_x^{2k}(\mathbb{R} \times \mathbb{R}^n)} \leq \left(C_{n,k} \int_{\mathbb{R}^{nk}} |\widehat{F}(\eta)|^2 K(\eta)^{\frac{n(k-1)-2}{2}} d\eta \right)^{1/2k}, \quad (2)$$

with

$$C_{n,k} = \left[2^{n(k-1)-1} k^{n/2} \pi^{(n(k-1)-2)/2} \Gamma\left(\frac{n(k-1)}{2}\right) \right]^{-1}. \quad (3)$$

On the right hand side of (2) we write $\eta \in \mathbb{R}^{nk}$ as $\eta = (\eta_1, \eta_2, \dots, \eta_k)$ with each $\eta_j \in \mathbb{R}^n$; $F(\eta) = f(\eta_1)f(\eta_2)\dots f(\eta_k)$; and the kernel

$$K(\eta) = \frac{1}{k} \sum_{1 \leq i < j \leq k} |\eta_i - \eta_j|^2.$$

Equality occurs if and only if f is a Gaussian.

Remarks

- The Fourier transform of the function $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is given by

$$\widehat{f}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\omega \cdot x} f(x) \, dx.$$

- The solution of (1) can be given in terms of the Fourier transform

$$u(t, x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \omega} e^{-it|\omega|^2} \widehat{f}(\omega) \, d\omega.$$

- Here we shall refer as Gaussians the functions of the form

$$f(x) = e^{A|x|^2 + b \cdot x + C},$$

where $A, C \in \mathbb{C}$, $b \in \mathbb{C}^n$ and $\Re(A) < 0$.

Applications

Corollary (Sharp Strichartz in low dimensions)

In dimension $n = 1$ we have

$$\|u(t, x)\|_{L_t^6 L_x^6(\mathbb{R} \times \mathbb{R})} \leq 12^{-1/12} \|f\|_{L^2(\mathbb{R})}, \quad (4)$$

and

$$\|u(t, x)\|_{L_t^8 L_x^4(\mathbb{R} \times \mathbb{R})} \leq 2^{-1/4} \|f\|_{L^2(\mathbb{R})}. \quad (5)$$

In dimension $n = 2$ we have

$$\|u(t, x)\|_{L_t^4 L_x^4(\mathbb{R} \times \mathbb{R}^2)} \leq 2^{-1/2} \|f\|_{L^2(\mathbb{R}^2)}. \quad (6)$$

These inequalities are sharp and equality occurs if and only if f is a Gaussian.

Applications

Corollary (Sharp Sobolev-Strichartz in low dimensions)

We have

$$\|u(t, x)\|_{L_t^{10}L_x^{10}(\mathbb{R}\times\mathbb{R})} \leq (2\sqrt{5}\pi)^{-1/10} \|f'\|_{L^2(\mathbb{R})}^{1/5} \|f\|_{L^2(\mathbb{R})}^{4/5}, \quad (7)$$

$$\|u(t, x)\|_{L_t^{12}L_x^6(\mathbb{R}\times\mathbb{R})} \leq (6\pi)^{-1/12} \|f'\|_{L^2(\mathbb{R})}^{1/6} \|f\|_{L^2(\mathbb{R})}^{5/6}, \quad (8)$$

$$\|u(t, x)\|_{L_t^{16}L_x^4(\mathbb{R}\times\mathbb{R})} \leq (8\pi)^{-1/16} \|f'\|_{L^2(\mathbb{R})}^{1/8} \|f\|_{L^2(\mathbb{R})}^{7/8}, \quad (9)$$

$$\|u(t, x)\|_{L_t^6L_x^6(\mathbb{R}\times\mathbb{R}^2)} \leq (12\pi)^{-1/6} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{1/3} \|f\|_{L^2(\mathbb{R}^2)}^{2/3}, \quad (10)$$

$$\|u(t, x)\|_{L_t^8L_x^4(\mathbb{R}\times\mathbb{R}^2)} \leq (16\pi)^{-1/8} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{1/4} \|f\|_{L^2(\mathbb{R}^2)}^{3/4}, \quad (11)$$

$$\|u(t, x)\|_{L_t^4L_x^4(\mathbb{R}\times\mathbb{R}^4)} \leq (32\pi)^{-1/4} \|\nabla f\|_{L^2(\mathbb{R}^4)}^{1/2} \|f\|_{L^2(\mathbb{R}^4)}^{1/2}. \quad (12)$$

These inequalities are sharp and equality occurs if and only if f is a Gaussian.

Restriction Estimates

The Schrödinger and wave equations are related to the restriction problem for the paraboloid and cone, respectively,

$$S_{\text{parab}} := \{(\tau, \omega) \in \mathbb{R} \times \mathbb{R}^n : \tau = |\omega|^2\},$$

and

$$S_{\text{cone}} := \{(\tau, \omega) \in \mathbb{R} \times \mathbb{R}^n : \tau = |\omega|\}.$$

We endow these surfaces $S \subset \mathbb{R}^{n+1}$ with canonical measures $d\sigma$ given by

$$\int_{S_{\text{parab}}} g(\tau, \omega) d\sigma = \int_{\mathbb{R}^n} g(|\omega|^2, \omega) d\omega,$$

and

$$\int_{S_{\text{cone}}} g(\tau, \omega) d\sigma = \int_{\mathbb{R}^n} g(|\omega|, \omega) \frac{d\omega}{|\omega|}.$$

Restriction estimates (cont.)

In this setting, the restriction estimates are a priori inequalities of the form

$$\|\widehat{h}|_S\|_{L^q(S; d\sigma)} \leq C_{p,q,S} \|h\|_{L^p(\mathbb{R}^{n+1})}. \quad (13)$$

Current state of the problem:

	Dilation Inv.	Restriction Conj.	Best Result
Parab	$q = \frac{np'}{n+2}$	$p' > \frac{2n+2}{n}$	$p' > \frac{2n+6}{n+1}$ (Tao '03)
Cone	$q = \frac{(n-1)p'}{n+1}$	$p' > \frac{2n}{n-1}$	$p' > \frac{2n+6}{n+1}$ (Wolff '00)

Sharp Restriction Estimates

Theorem

For the paraboloid we have

$$\|\widehat{h}|_S\|_{L^2(S; d\sigma)} \leq (2\pi)^{-1/2} 2^{-1/12} \|h\|_{L^{6/5}(\mathbb{R}^2)},$$

and

$$\|\widehat{h}|_S\|_{L^2(S; d\sigma)} \leq (4\pi)^{-1/2} \|h\|_{L^{4/3}(\mathbb{R}^3)}.$$

Theorem

For the cone we have

$$\|\widehat{h}|_S\|_{L^2(S; d\sigma)} \leq (2\pi)^{1/3} \|h\|_{L^{6/5}(\mathbb{R}^3)},$$

and

$$\|\widehat{h}|_S\|_{L^2(S; d\sigma)} \leq (2\pi)^{1/4} \|h\|_{L^{4/3}(\mathbb{R}^4)}.$$

Maximizers characterized via the dual extension estimates.

Proof of the Main Theorem

Recall what we want prove:

For $k \in \mathbb{Z}$, $k \geq 2$ and $(n, k) \neq (1, 2)$ we have

$$\|u(t, x)\|_{L_t^{2k} L_x^{2k}(\mathbb{R} \times \mathbb{R}^n)} \leq \left(C_{n,k} \int_{\mathbb{R}^{nk}} |\widehat{F}(\eta)|^2 K(\eta)^{\frac{n(k-1)-2}{2}} d\eta \right)^{1/2k},$$

with

$$C_{n,k} = \left[2^{n(k-1)-1} k^{n/2} \pi^{(n(k-1)-2)/2} \Gamma\left(\frac{n(k-1)}{2}\right) \right]^{-1}.$$

On the right hand side we write $\eta \in \mathbb{R}^{nk}$ as $\eta = (\eta_1, \eta_2, \dots, \eta_k)$ with each $\eta_i \in \mathbb{R}^n$; $F(\eta) = f(\eta_1)f(\eta_2)\dots f(\eta_k)$; and the kernel

$$K(\eta) = \frac{1}{k} \sum_{1 \leq i < j \leq k} |\eta_i - \eta_j|^2.$$

Representation Lemma

- Define $F_1(\eta) = \widehat{F}(\eta)K(\eta)^{\frac{n(k-1)-2}{4}}$.
- Let $E \subset L^2(\mathbb{R}^{nk})$ be the closed subspace consisting of the functions invariant under any orthonormal transformation R fixing the vectors $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}^{nk}$ given by

$$\alpha_j = (\mathbf{e}_j, \mathbf{e}_j, \dots, \mathbf{e}_j) \quad (k \text{ times}), \quad (\star)$$

where $\mathbf{e}_j = (0, 0, \dots, 1, \dots, 0)$ is the i -th canonical vector in \mathbb{R}^n . Denote by $P_E : L^2(\mathbb{R}^{nk}) \rightarrow L^2(\mathbb{R}^{nk})$ the projection operator.

Lemma

$$\int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} dx dt = C_{n,k} \langle P_E(F_1), F_1 \rangle_{L^2(\mathbb{R}^{nk})}.$$

Assuming the Lemma

$$\begin{aligned}\int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} dx dt &= C_{n,k} \langle P_E(F_1), F_1 \rangle_{L^2(\mathbb{R}^{nk})} \\ &\leq C_{n,k} \|F_1\|_{L^2(\mathbb{R}^{nk})}^2 \\ &= C_{n,k} \int_{\mathbb{R}^{nk}} |\widehat{F}(\eta)|^2 K(\eta)^{\frac{n(k-1)-2}{2}} d\eta.\end{aligned}\tag{14}$$

A necessary and sufficient condition for equality in (14) is:

$$\widehat{F}(\eta) K(\eta)^{\frac{n(k-1)-2}{4}} = F_1(x) \in E$$

Note: The kernel $K(\eta)$ satisfies $(\star) \Rightarrow \widehat{F}(\eta)$ must satisfy (\star) .

Proof of the Representation Lemma

Start with $u(t, x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \omega} e^{-it|\omega|^2} \widehat{f}(\omega) d\omega$. Then

$$|u(t, x)|^{2k} = \frac{1}{(2\pi)^{nk}} \int_{\mathbb{R}^{nk} \times \mathbb{R}^{nk}} e^{ix \cdot (\sum \eta_i - \sum \xi_i)} e^{-it(|\eta|^2 - |\xi|^2)} \widehat{F}(\eta) \overline{\widehat{F}(\xi)} d\eta d\xi,$$

where $\eta = (\eta_1, \eta_2, \dots, \eta_k)$ and $\xi = (\xi_1, \xi_2, \dots, \xi_k)$, each η_i and ξ_i in \mathbb{R}^n .

Fact: $\delta_n(w) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix \cdot w} dx$. Then

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} dx dt \\ &= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk} \times \mathbb{R}^{nk}} \delta_n \left(\sum_{i=1}^k \eta_i - \sum_{i=1}^k \xi_i \right) \delta(|\eta|^2 - |\xi|^2) \widehat{F}(\eta) \overline{\widehat{F}(\xi)} d\eta d\xi \\ &= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk} \times \mathbb{R}^{nk}} \left(\prod_{i=1}^n \delta((\eta - \xi) \cdot \alpha_i) \right) \delta(|\eta|^2 - |\xi|^2) \widehat{F}(\eta) \overline{\widehat{F}(\xi)} d\eta d\xi. \end{aligned}$$

Proof of the Representation Lemma

Start with $u(t, x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \omega} e^{-it|\omega|^2} \widehat{f}(\omega) d\omega$. Then

$$|u(t, x)|^{2k} = \frac{1}{(2\pi)^{nk}} \int_{\mathbb{R}^{nk} \times \mathbb{R}^{nk}} e^{ix \cdot (\sum \eta_i - \sum \xi_i)} e^{-it(|\eta|^2 - |\xi|^2)} \widehat{F}(\eta) \overline{\widehat{F}(\xi)} d\eta d\xi,$$

where $\eta = (\eta_1, \eta_2, \dots, \eta_k)$ and $\xi = (\xi_1, \xi_2, \dots, \xi_k)$, each η_i and ξ_i in \mathbb{R}^n .

Fact: $\delta_n(w) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix \cdot w} dx$. Then

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} dx dt \\ &= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk} \times \mathbb{R}^{nk}} \delta_n \left(\sum_{i=1}^k \eta_i - \sum_{i=1}^k \xi_i \right) \delta(|\eta|^2 - |\xi|^2) \widehat{F}(\eta) \overline{\widehat{F}(\xi)} d\eta d\xi \\ &= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk} \times \mathbb{R}^{nk}} \left(\prod_{i=1}^n \delta((\eta - \xi) \cdot \alpha_i) \right) \delta(|\eta|^2 - |\xi|^2) \widehat{F}(\eta) \overline{\widehat{F}(\xi)} d\eta d\xi. \end{aligned}$$

Proof of the Representation Lemma (cont.)

Rewrite the last expression in the strategic way:

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} dx dt \\ &= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{(\mathbb{R}^{nk})^2} \frac{(\prod_{i=1}^n \delta((\eta - \xi) \cdot \alpha_i)) \delta(|\eta|^2 - |\xi|^2)}{(K(\eta)K(\xi))^{\frac{n(k-1)-2}{4}}} F_1(\eta) \overline{F_1(\xi)} d\eta d\xi \end{aligned}$$

Define the (self-adjoint) operator

$$AG(\xi) = \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk}} \frac{(\prod_{i=1}^n \delta((\eta - \xi) \cdot \alpha_i)) \delta(|\eta|^2 - |\xi|^2)}{(K(\eta)K(\xi))^{\frac{n(k-1)-2}{4}}} G(\eta) d\eta.$$

In this setting we have

$$\int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} dx dt = \langle AF_1, F_1 \rangle_{L^2(\mathbb{R}^{nk})}.$$

Goal: Prove that $A = C_{n,k} P_E$.

Proof of the Representation Lemma (cont.)

Rewrite the last expression in the strategic way:

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} dx dt \\ &= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{(\mathbb{R}^{nk})^2} \frac{(\prod_{i=1}^n \delta((\eta - \xi) \cdot \alpha_i)) \delta(|\eta|^2 - |\xi|^2)}{(K(\eta)K(\xi))^{\frac{n(k-1)-2}{4}}} F_1(\eta) \overline{F_1(\xi)} d\eta d\xi \end{aligned}$$

Define the (self-adjoint) operator

$$AG(\xi) = \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk}} \frac{(\prod_{i=1}^n \delta((\eta - \xi) \cdot \alpha_i)) \delta(|\eta|^2 - |\xi|^2)}{(K(\eta)K(\xi))^{\frac{n(k-1)-2}{4}}} G(\eta) d\eta.$$

In this setting we have

$$\int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} dx dt = \langle AF_1, F_1 \rangle_{L^2(\mathbb{R}^{nk})}.$$

Goal: Prove that $A = C_{n,k} P_E$.

Proof of the Representation Lemma (cont.)

Write

$$AG(\xi) = C_{n,k} \int_{\mathbb{R}^{nk}} G(\eta) m_{\xi}(d\eta),$$

with

$$m_{\xi}(d\eta) = \frac{k^{n/2} \Gamma\left(\frac{n(k-1)}{2}\right)}{\pi^{n(k-1)/2}} \frac{(\prod_{i=1}^n \delta((\eta - \xi) \cdot \alpha_i)) \delta(|\eta|^2 - |\xi|^2)}{(K(\eta)K(\xi))^{\frac{n(k-1)-2}{4}}} d\eta$$

Lemma

- (i) For all $\xi \in \mathbb{R}^{nk}$ the measure $m_{\xi}(d\eta)$ is a probability measure on \mathbb{R}^{nk} .
- (ii) For all Borel measurable sets $B \subset \mathbb{R}^{nk}$, we have

$$\int_{\mathbb{R}^{nk}} m_{\xi}(B) d\xi = |B|,$$

where $|B|$ denotes the Lebesgue measure of B .

Proof of the Representation Lemma (cont.)

Write

$$AG(\xi) = C_{n,k} \int_{\mathbb{R}^{nk}} G(\eta) m_{\xi}(d\eta),$$

with

$$m_{\xi}(d\eta) = \frac{k^{n/2} \Gamma\left(\frac{n(k-1)}{2}\right)}{\pi^{n(k-1)/2}} \frac{(\prod_{i=1}^n \delta((\eta - \xi) \cdot \alpha_i)) \delta(|\eta|^2 - |\xi|^2)}{(K(\eta)K(\xi))^{\frac{n(k-1)-2}{4}}} d\eta$$

Lemma

- (i) For all $\xi \in \mathbb{R}^{nk}$ the measure $m_{\xi}(d\eta)$ is a probability measure on \mathbb{R}^{nk} .
- (ii) For all Borel measurable sets $B \subset \mathbb{R}^{nk}$, we have

$$\int_{\mathbb{R}^{nk}} m_{\xi}(B) d\xi = |B|,$$

where $|B|$ denotes the Lebesgue measure of B .

Proof of the Representation Lemma (cont.)

- Previous Lemma + Jensen's Inequality:

$$\|AG\|_{L^2(\mathbb{R}^{nk})} \leq C_{n,k} \|G\|_{L^2(\mathbb{R}^{nk})}.$$

- The fact that

$$\begin{aligned} K(\eta) &= \frac{1}{k} \sum_{1 \leq i < j \leq k} |\eta_i - \eta_j|^2 \\ &= |\eta|^2 - \frac{|\eta_1 + \eta_2 + \dots + \eta_k|^2}{k} = |\eta|^2 - \frac{\sum_{i=1}^n (\eta \cdot \alpha_i)^2}{k}. \end{aligned}$$

implies that

$$AG(R\xi) = AG(\xi),$$

so A maps $L^2(\mathbb{R}^{nk})$ onto E .

Proof of the Representation Lemma (cont.)

- A is self-adjoint $\Rightarrow A(E^\perp) = 0$.
- Finally $A = C_{n,k}I$ on E . To see this just test A on the dense subset of functions

$$G(\eta) = H(\eta \cdot \alpha_1, \eta \cdot \alpha_2, \dots, \eta \cdot \alpha_n, |\eta|^2),$$

where $H \in C_0^\infty(\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} \times \mathbb{R}^+)$.

Conclusion: $A = C_{n,k}P_E$.

Gaussian Maximizers

A necessary and sufficient condition for equality is:

$$\widehat{F}(\eta)K(\eta)^{\frac{n(k-1)-2}{4}} = F_1(x) \in E$$

The kernel $K(\eta)$ satisfies $(\star) \Rightarrow \widehat{F}(\eta) = \widehat{f}(\eta_1)\dots\widehat{f}(\eta_k)$ must satisfy (\star) .

Step 1. Let $g : \mathbb{R}^n \rightarrow \mathbb{C}$ be a measurable function such that $G(\eta) = g(\eta_1)g(\eta_2)\dots g(\eta_k)$ satisfies

$$\int_{\mathbb{R}^{nk}} |G(\eta)|^2 K(\eta)^{\frac{n(k-1)-2}{2}} d\eta < \infty.$$

Then $g \in L^p(\mathbb{R}^n)$ for $p = \frac{2nk}{2nk-n-2}$.

This is a consequence of reversed HLS inequality (Beckner) and interpolation.

Gaussian Maximizers (cont.)

Step 2. Let $g \in L^p(\mathbb{R}^n)$ be such that $G(\eta) = g(\eta_1)g(\eta_2)\dots g(\eta_k)$ satisfies the property (\star) . Then g is a product of one-dimensional functions.

Proof. Write $\eta_i = (\eta_{i1}, \eta_{i2}, \dots, \eta_{in})$, for $i = 1, 2, \dots, k$. If $g \neq 0$, there exists a cube $J = \prod_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n$ such that

$$\int_J g(y) dy = A \neq 0.$$

Consider the orthonormal transformation R in \mathbb{R}^{nk} that switches the coordinates η_{11} and η_{21} on $\eta = (\eta_1, \dots, \eta_k)$. Then

$$\begin{aligned} g(\eta_{11}, \eta_{12}, \dots, \eta_{1n})g(\eta_{21}, \eta_{22}, \dots, \eta_{2n})g(\eta_3)\dots g(\eta_k) \\ = g(\eta_{21}, \eta_{12}, \dots, \eta_{1n})g(\eta_{11}, \eta_{22}, \dots, \eta_{2n})g(\eta_3)\dots g(\eta_k). \end{aligned}$$

Integrate the last expression with respect to $d\eta_2 d\eta_3 \dots d\eta_k$ on $J \times J \times \dots \times J$ to find:

Gaussian Maximizers (cont.)

$$\begin{aligned} & A^{k-1} g(\eta_{11}, \eta_{12}, \dots, \eta_{1n}) \\ &= A^{k-2} \int_{a_1}^{b_1} g(\eta_{21}, \eta_{12}, \dots, \eta_{1n}) d\eta_{21} \int_{J'} g(\eta_{11}, \eta_{22}, \dots, \eta_{2n}) d\eta'_2, \end{aligned} \quad (15)$$

where $J' = \prod_{i=2}^n [a_i, b_i]$ and $d\eta'_2 = d\eta_{22}d\eta_{23}\dots d\eta_{2n}$. Expression (15) plainly says that

$$g(\eta_{11}, \eta_{12}, \dots, \eta_{1n}) = w_1(\eta_{11}) h_1(\eta_{12}, \dots, \eta_{1n}).$$

By repeating this argument we arrive at

$$g(\eta_{11}, \eta_{12}, \dots, \eta_{1n}) = g_1(\eta_{11})g_2(\eta_{12})\dots g_n(\eta_{1n}).$$

Gaussian Maximizers (cont.)

From the fact that $G(R(\theta)\eta) = G(\eta)$ and $R(0) = I$ we obtain

$$0 = -2 \frac{\partial G(R(\theta)\eta)}{\partial \theta} \Big|_{\theta=0} = \\ [(\eta_{12} - \eta_{22})\partial_{\eta_{11}} - (\eta_{11} - \eta_{21})\partial_{\eta_{12}} - (\eta_{12} - \eta_{22})\partial_{\eta_{21}} + (\eta_{11} - \eta_{21})\partial_{\eta_{22}}] G(\eta).$$

By introducing the logarithmic derivatives $h'_i = g'_i/g_i$ we obtain

$$(\eta_{12} - \eta_{22})h'_1(\eta_{11}) - (\eta_{11} - \eta_{21})h'_2(\eta_{12}) \\ - (\eta_{12} - \eta_{22})h'_1(\eta_{21}) + (\eta_{11} - \eta_{21})h'_2(\eta_{22}) = 0.$$

Differentiating on η_{11} , and then on η_{22} , we obtain

$$h''_1(\eta_{11}) = h''_2(\eta_{22}),$$

Repeat the argument for γ_1 and γ_j yielding $h''_1 = h''_j = C$ for all $j = 1, 2, \dots, n$. This proves that all g_i 's are Gaussians with the same covariance, and thus g will itself be a Gaussian.

Gaussian Maximizers (cont.)

To reduce matters to the smooth non-vanishing case we use a classical argument due to Carlen. Let Q_ϵ the convolution with the Gaussian kernel on \mathbb{R}^n

$$\phi_\epsilon(y) = \frac{1}{(2\pi\epsilon)^{n/2}} e^{-\frac{|y|^2}{2\epsilon}}.$$

Step 4. Let $g \in L^p(\mathbb{R}^n)$ be such that $G(\eta)$ satisfies the property (\star) . Assume $Q_\epsilon(g)$ never vanishes as $\epsilon \rightarrow 0$. Then g is a Gaussian.

Step 5. Let $g \in L^p(\mathbb{R}^n)$ be such that $G(\eta)$ satisfies the property (\star) . Then $Q_\epsilon(g)$ never vanishes as $\epsilon \rightarrow 0$.

Conclusion: Steps 1- 5 with $g = \widehat{f}$

$\Rightarrow \widehat{f}$ is a Gaussian

$\Rightarrow f$ is a Gaussian.

Thank you!



E. Carlen,

Superadditivity of Fisher's information and logarithmic Sobolev inequalities,

[J. Funct. Anal. 101 \(1991\), no. 1, 194–211.](#)



D. Foschi,

Maximizers for the Strichartz inequality,

[J. Eur. Math. Soc. \(JEMS\) 9 \(2007\), no. 4, 739–774.](#)



D. Hundertmark and V. Zharnitsky,

On sharp Strichartz inequalities in low dimensions,

[Int. Math. Res. Not. 2006, Art. ID 34080, 18 pp.](#)



M. Kunze,

On the existence of a maximizer for the Strichartz inequality,

[Comm. Math. Phys. 243 \(2003\), no. 1, 137–162.](#)



R. Strichartz,

Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations,

[Duke Math. J. 44 \(1977\), 705–774.](#)



S. Shao,

Maximizers for the Strichartz inequalities and the Sobolev-Strichartz inequalities for the Schrödinger equation, preprint [arXiv:0809.0153](https://arxiv.org/abs/0809.0153).



T. Tao,

A sharp bilinear restriction estimate on paraboloids, *Geom. Funct. Anal.* 13 (2003), no. 6, 1359–1384.



T. Wolff,

A sharp bilinear cone restriction estimate, *Annals of Math.* 153 (2001), 661–698.