Regular components of moduli spaces of stable maps

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1 Introduction

The purpose of this note is to prove the existence of 'nice' components of the Hilbert scheme of curves $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ of genus $g \geq 2$ and bidegree (k, d). We can also phrase our result using the Kontsevich moduli space of stable maps to $\mathbb{P}^1 \times \mathbb{P}^r$. We work over an algebraically closed field of characteristic zero.

For a smooth projective variety Y and a class $\beta \in H_2(Y, \mathbb{Z})$, one considers the moduli stack $\overline{\mathcal{M}}_g(Y, \beta)$ of stable maps $f : C \to Y$, with C a reduced connected nodal curve of genus g and $f_*([C]) = \beta$ (see [FP] for the construction of these stacks). The open substack $\mathcal{M}_g(Y, \beta)$ of $\overline{\mathcal{M}}_g(Y, \beta)$ parametrizes maps from smooth curves to Y. By $\overline{M}_g(Y, \beta)$ we denote the coarse moduli space corresponding to the stack $\overline{\mathcal{M}}_g(Y, \beta)$ and similarly $\overline{M}_g$ is the moduli space corresponding to the stack $\overline{\mathcal{M}}_g$ of stable curves of genus g. We denote by $\pi : \overline{\mathcal{M}}_g(Y, \beta) \to \overline{\mathcal{M}}_g$ the natural projection. The *expected dimension* of the stack $\overline{\mathcal{M}}_g(Y, \beta)$ is

$$\chi(g, Y, \beta) = \dim(Y)\ (1 - g) + 3g - 3 - \beta \cdot K_Y.$$

Since in general the geometry of $\overline{\mathcal{M}}_g(Y, \beta)$ is quite messy (e.g. existence of many components, some nonreduced and/or not of expected dimension), it is not obvious what the definition of a nice component of $\overline{\mathcal{M}}_g(Y, \beta)$ should be. Following Sernesi [Se] we introduce the following terminology:

Definition. A component V of $\overline{\mathcal{M}}_g(Y, \beta)$ is said to be *regular* if it is generically smooth and of dimension $\chi(g, Y, \beta)$. We say that V has the *expected number of moduli* if

$$\dim \pi(V) = \min\big(3g - 3, \chi(g, Y, \beta) - \dim \mathrm{Aut}(Y)\big).$$

In this paper we only construct regular components of moduli spaces of stable maps. We study the stacks $\overline{\mathcal{M}}_g(Y, \beta)$ when $Y = \mathbb{P}^1 \times \mathbb{P}^r$, $r \geq 3$ and $\beta = (k, d) \in H_2(\mathbb{P}^1 \times \mathbb{P}^r, \mathbb{Z})$. We denote by $\rho(g, r, d) = g - (r + 1)(g - d + r)$ the *Brill-Noether number* governing the existence of $\mathfrak{g}^r_d$'s on curves of genus g. Our main result is the following:

Theorem 1 *Let g, r, d and k be positive integers with $r \geq 3, \rho(g, r, d) < 0$ and*

$$(2 - \rho(g, r, d))r + 2 \leq k \leq (g + 2)/2.$$

Then there exists a regular component of the stack of maps $\mathcal{M}_g\big(\mathbb{P}^1 \times \mathbb{P}^r, (k, d)\big)$.

We introduce the *Brill-Noether locus* $M^r_{g,d} = \{[C] \in M_g : C \text{ has a } \mathfrak{g}^r_d\}$, in the case $\rho(g,r,d) < 0$. The expected codimension of $M^r_{g,d}$ inside M_g is $-\rho(g,r,d)$. We view Theorem 1 as a tool in the study of the relative position of the loci $M^1_{g,k}$ and $M^r_{g,d}$ when $r \geq 3, \rho(g,1,k) < 0 \Leftrightarrow k < (g+2)/2$ and $\rho(g,r,d) < 0$. The stack $\mathcal{M}_g\big(\mathbb{P}^1 \times \mathbb{P}^r, (k,d)\big)$ comes naturally into play when looking at the intersection in M_g of the loci $M^1_{g,k}$ and $M^r_{g,d}$. In such a setting, if V is a regular component of $M_g\big(\mathbb{P}^1 \times \mathbb{P}^r, (k,d)\big)$, then $M^1_{g,k}$ and $M^r_{g,d}$ intersect properly along $\pi(V)$. It is very plausible that one has a similar statement to Theorem 1 when $\rho(g,r,d) \geq 0$ and/or $\rho(g,1,k) \geq 0$, but from our perspective that seems of less interest because it would be essentially a statement about linear series on the general curve of genus g with no implications on the problem of understanding the geography of the Brill-Noether loci inside M_g.

Regarding the problem of existence of regular components of $\mathcal{M}_g(Y,\beta)$, so far the spaces $\mathcal{M}_g(\mathbb{P}^r,d)$ have received the bulk of attention. When $r=1,2$ the problem boils down to the study of the Hurwitz scheme and of the Severi variety of plane curves which are known to be irreducible and regular. For $r \geq 3$ we have the following result of Sernesi (cf. [Se, p. 26]):

Proposition 1.1 *For all g,r,d such that $d \geq r+1$ and*

$$-\frac{g}{r} + \frac{r+1}{r} \leq \rho(g,r,d) < 0,$$

there exists a regular component V of $\mathcal{M}_g(\mathbb{P}^r,d)$ which has the expected number of moduli. A general point of V corresponds to an embedding $C \hookrightarrow \mathbb{P}^r$ by a complete linear system (i.e. $h^0(C,\mathcal{O}_C(1)) = r+1$), the normal bundle N_C satisfies $H^1(C,N_C) = 0$ and the Petri map

$$\mu_0(C) : H^0(C,\mathcal{O}_C(1)) \otimes H^0(C,K_C(-1)) \to H^0(C,K_C)$$

is surjective.

A. Lopez has obtained significant improvements on the range of g,r,d such that there exists a regular component of $\mathcal{M}_g(\mathbb{P}^r,d)$: if $h(r) = (4r^3+8r^2-9r+3)/(r+3)$, then for all g,r,d such that $-(2-6/(r+3))g + h(r) \leq \rho(g,r,d) < 0$ there exists a regular component of $\mathcal{M}_g(\mathbb{P}^r,d)$ with the expected number of moduli (cf. [Lo]).

When Y is a smooth surface, methods from [AC] can be employed to show that if V is a component of $\mathcal{M}_g(Y,\beta)$ with $\dim(V) \geq g+1$ and which contains a point $[f : C \to Y]$ with $\deg(f) = 1$ (i.e. f is generically injective), then V is regular. Here it is crucial that the normal sheaf N_f is of rank 1 as then the Clifford Theorem provides an easy criterion for the vanishing of $H^1(C,N_f)$, which turns out to be a sufficient criterion for regularity (see Section 2).

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2 Deformations of maps and smoothings of space curves

We review some facts about deformations of maps and smoothings of reducible nodal curves in $\mathbb{P}^r$. Our references are [Ran] and [Se].

We start by describing the deformation theory of maps between complex algebraic varieties when the source is (possibly) singular and the target is smooth. Let $f : X \to Y$ be a morphism between complex projective varieties, with Y being smooth. We denote by $\mathrm{Def}(X, f, Y)$ the space of first-order deformations of the map f when X and Y are not considered fixed. The space of first-order deformations of X (resp. Y) is denoted by $\mathrm{Def}(X)$ (resp. $\mathrm{Def}(Y)$). We have the standard identification $\mathrm{Def}(X) = \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$. The deformation space $\mathrm{Def}(X, f, Y)$ fits in the following exact sequence:

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) \longrightarrow \mathrm{Def}(X, f, Y) \longrightarrow \mathrm{Def}(X) \oplus \mathrm{Def}(Y) \longrightarrow \mathrm{Ext}^1_f(\Omega_Y, \mathcal{O}_X). \quad (1)$$

The second arrow is given by the natural forgetful maps, the space $\mathrm{Hom}_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) = H^0(X, f^*T_Y)$ parametrizes first-order deformations of $f : X \to Y$ when both X and Y are fixed, while for A, B, respectively $\mathcal{O}_X$ and $\mathcal{O}_Y$-modules, $\mathrm{Ext}^i_f(B, A)$ denotes the derived functor of $\mathrm{Hom}_f(B, A) = \mathrm{Hom}_{\mathcal{O}_X}(f^*B, A) = \mathrm{Hom}_{\mathcal{O}_Y}(B, f_*A)$. Under reasonable assumptions (trivially satisfied when f is a finite map between nodal curves) one has that $\mathrm{Ext}^1_f(\Omega_Y, \mathcal{O}_X) = \mathrm{Ext}^1(f^*\Omega_Y, \mathcal{O}_X)$. Using (1) it follows that when X is smooth and irreducible and Y is rigid (e.g. a product of projective spaces) $\mathrm{Def}(X, f, Y) = H^0(X, N_f)$, where $N_f = Coker\{T_X \to f^*T_Y\}$ is the normal sheaf of the map f.

For a smooth variety Y, a class $\beta \in H_2(Y, \mathbb{Z})$ and a point $[f : C \to Y] \in \mathcal{M}_g(Y, \beta)$ we have that $T_{[f]}(\overline{\mathcal{M}}_g(Y, \beta)) = H^0(C, N_f)$. If moreover $\deg(f) = 1$ and $H^1(C, N_f) = 0$, then every class in $H^0(C, N_f)$ is unobstructed, f is an immersion (cf. [AC, Lemma 1.4]) and $\overline{\mathcal{M}}_g(Y, \beta)$ is smooth and of the expected dimension at the point $[f]$, that is, $[f]$ belongs to a regular component of $\overline{\mathcal{M}}_g(Y, \beta)$.

Let $C \subseteq \mathbb{P}^r$ be a stable curve of genus g and degree d. If $\mathcal{I}_C$ is the ideal sheaf of C we denote by $N_C := Hom(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ the normal sheaf of C in $\mathbb{P}^r$. Assume that $H^1(C, N_C) = 0$ and that $h^0(C, \mathcal{O}_C(1)) = r + 1$, that is, C is embedded by a complete linear system. The differential of the map $\pi : \overline{\mathcal{M}}_g(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_g$ at the point $[C \hookrightarrow \mathbb{P}^r]$ is given by the natural map $H^0(C, N_C) \to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$. If ω_C denotes the dualizing sheaf of C, then $\mathrm{rk}(d\pi)_{[C \hookrightarrow \mathbb{P}^r]} = 3g - 3 - \dim \mathrm{Ker}\mu_0(C)$, where

$$\mu_0(C) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \to H^0(C, \omega_C)$$

is the Petri map. In particular $(d\pi)_{[C \hookrightarrow \mathbb{P}^r]}$ has rank $3g - 3 + \rho(g, r, d)$ if and only if $\mu_0(C)$ is surjective.

In the same setting, via the standard identification $T_{[C]}(\overline{\mathcal{M}}_g)^\vee = H^0(C, \omega_C \otimes \Omega_C)$, the annihilator $(\mathrm{Im}(d\pi)_{[C \hookrightarrow \mathbb{P}^r]})^\perp \subseteq H^0(C, \omega_C \otimes \Omega_C)$ can be naturally identified with $\mathrm{Im}(\mu_1(C))$, where

$$\mu_1(C) : \mathrm{Ker}\mu_0(C) \to H^0(C, \Omega_C \otimes \omega_C)$$

is the Gaussian map obtained from taking the 'derivative' of $\mu_0(C)$ (cf. [CGGH, p. 163]).

In Section 3 we will smooth curves $X \subseteq \mathbb{P}^r$ which are unions of two smooth curves C and E meeting quasi-transversally (i.e. having distinct tangent lines) at a finite set Δ. For such a curve one has the exact sequences (cf. [Se, p. 35])

$$0 \longrightarrow \mathcal{O}_E(-\Delta) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0, \tag{2}$$

and

$$0 \longrightarrow \Omega_E \longrightarrow \omega_X \longrightarrow \Omega_C(\Delta) \longrightarrow 0. \tag{3}$$

Also in Section 3 we will use an inductive procedure to construct curves $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ with $H^1(C, N_{C/\mathbb{P}^1\times\mathbb{P}^r}) = 0$. The induction step uses the following result (cf. [BE, Lemma 2.3]):

Proposition 2.1 *Let $C \subseteq \mathbb{P}^r$ be a smooth curve with $H^1(C, N_C) = 0$. We take $r+2$ points $p_1, \dots, p_{r+2} \in C$ in general linear position and a smooth rational curve $E \subseteq \mathbb{P}^r$ of degree r which meets C quasi-transversally at $p_1, \dots, p_{r+2}$. Then $X = C \cup E$ is smoothable in $\mathbb{P}^r$ and $H^1(X, N_X) = 0$.*

3 Existence of regular components of $\mathcal{M}_g\big(\mathbb{P}^1\times\mathbb{P}^r, (k,d)\big)$

In this section we prove the existence of regular components of $\overline{\mathcal{M}}_g\big(\mathbb{P}^1 \times \mathbb{P}^r, (k,d)\big)$ in the case $k \geq r+2, d \geq r \geq 3$, and $\rho(g,r,d) < 0$. We achieve this by constructing smooth curves $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ of bidegree (k,d) satisfying $H^1(C, N_{C/\mathbb{P}^1\times\mathbb{P}^r}) = 0$.

Let us fix integers $g \geq 2, d \geq r \geq 3$ and $k \geq 2$, as well as a smooth curve C of genus g with maps $f_1 : C \to \mathbb{P}^1, f_2 : C \to \mathbb{P}^r$, such that $\deg(f_1) = k,\ \deg(f_2(C)) = d$ and f_2 is generically injective. Let us denote by $f : C \to \mathbb{P}^1 \times \mathbb{P}^r$ the product map. As usual we denote by $G^r_d(C)$ the scheme parametrizing $\mathfrak{g}^r_d$'s on C.

There is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & T_C & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & T_C & \longrightarrow & f^*(T_{\mathbb{P}^1\times\mathbb{P}^r}) & \longrightarrow & N_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow = & & \downarrow & & \\
0 & \longrightarrow & T_C \oplus T_C & \longrightarrow & f_1^*(T_{\mathbb{P}^1}) \oplus f_2^*(T_{\mathbb{P}^r}) & \longrightarrow & N_{f_1} \oplus N_{f_2} & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}.$$

By taking cohomology in the last column, we see that the condition $H^1(C, N_f) = 0$ is equivalent to $H^1(C, N_{f_1}) = 0$ (automatic), $H^1(C, N_{f_2}) = 0$, and

$$\mathrm{Im}\{\delta_1 : H^0(C, N_{f_1}) \to H^1(C, T_C)\} + \mathrm{Im}\{\delta_2 : H^0(C, N_{f_2}) \to H^1(C, T_C)\} = H^1(C, T_C), \tag{4}$$

where δ_1 and δ_2 are coboundary maps. Condition (4) is equivalent (cf. Section 2) to

$$(d\pi_1)_{[f_1]}\ \big(T_{[f_1]}(\mathcal{M}_g(\mathbb{P}^1,k))\big)+(d\pi_2)_{[f_2]}\ \big(T_{[f_2]}(\mathcal{M}_g(\mathbb{P}^r,d))\big)=T_{[C]}(\mathcal{M}_g), \qquad (5)$$

where the projections $\pi_1:\mathcal{M}_g(\mathbb{P}^1,k)\to\mathcal{M}_g$ and $\pi_2:\mathcal{M}_g(\mathbb{P}^r,d)\to\mathcal{M}_g$ are the natural forgetful maps. Slightly abusing terminology, if C is a smooth curve and $(l_1,l_2)\in G^1_k(C)\times G^r_d(C)$ is a pair of base point free linear series on C, we say that (C,l_1,l_2) satisfies (5), if (C,f_1,f_2) satifies (5), where f_1 and f_2 are maps associated to l_1 and l_2.

Recall that a base point free pencil $\mathfrak{g}^1_k$ is said to be *simple* if the induced covering $f:C\to\mathbb{P}^1$ has a single ramification point x over each branch point and moreover $e_x(f)=2$.

We prove the existence of regular components of $\mathcal{M}_g\big(\mathbb{P}^1\times\mathbb{P}^r,(k,d)\big)$ using the following inductive procedure:

Proposition 3.1 *Fix positive integers g,r,d and k with $d\geq r\geq 3, k\geq r+2$ and $\rho(g,r,d)<0$. Let us assume that $C\subseteq\mathbb{P}^r$ is a smooth nondegenerate curve of degree d and genus g, such that $h^1(C,N_C)=0, h^0(C,\mathcal{O}_C(1))=r+1$ and the Petri map*

$$\mu_0(C)=\mu_0(C,\mathcal{O}_C(1)):H^0(C,\mathcal{O}_C(1))\otimes H^0(C,K_C(-1))\to H^0(C,K_C)$$

is surjective. Assume furthermore that C possesses a simple base point free pencil $\mathfrak{g}^1_k$ say l, such that $|\mathcal{O}_C(1)|(-l)=\emptyset$ and $(C,l,|\mathcal{O}_C(1)|)$ satisfies (5).

Then there exists a smooth nondegenerate curve $Y\subseteq\mathbb{P}^r$ with $g(Y)=g+r+1$, $\deg(Y)=d+r$ and a simple base point free pencil $l'\in G^1_k(Y)$, so that Y enjoys exactly the same properties: $h^1(Y,N_Y)=0$, $h^0(Y,\mathcal{O}_Y(1))=r+1$, the Petri map $\mu_0(Y)$ is surjective, $|\mathcal{O}_Y(1)|(-l')=\emptyset$ and $(Y,l',|\mathcal{O}_Y(1)|)$ satifies (5).

Proof. We first construct a reducible k-gonal nodal curve $X\subseteq\mathbb{P}^r$, with $p_a(X)=g+r+1$, $\deg(X)=d+r$, having all the required properties, then we prove that X can be smoothed in $\mathbb{P}^r$ preserving all properties we want.

Let $f_1:C\to\mathbb{P}^1$ be the degree k map corresponding to the pencil l. The covering f_1 is simple hence the monodromy of f_1 is the full symmetric group. Then since $|\mathcal{O}_C(1)|(-l)=\emptyset$, we have that for a general $\lambda\in\mathbb{P}^1$ the fibre $f_1^{-1}(\lambda)=p_1+\cdots+p_k$ consists of k distinct points in general linear position. Let $\Delta=\{p_1,\ldots,p_{r+2}\}$ be a subset of $f_1^{-1}(\lambda)$ and let $E\subseteq\mathbb{P}^r$ be a rational normal curve ($\deg(E)=r$) passing through $p_1,\ldots,p_{r+2}$. (Through any $r+3$ points in general linear position in $\mathbb{P}^r$, there passes a unique rational normal curve). We set $X:=C\cup E$, with C and E meeting quasi-transversally at Δ. Of course $p_a(X)=g+r+1$ and $\deg(X)=d+r$. Note that $\rho(g,r,d)=\rho(g+r+1,r,d+r)$.

We first prove that $[X]\in\overline{M}^1_{g+r+1,k}$ (that is, X is a limit of smooth k-gonal curves), by constructing an admissible covering of degree k having as domain a curve X', stably equivalent to X. Let $X':=X\cup D_{r+3}\cup\ldots\cup D_k$, where $D_i\simeq\mathbb{P}^1$ and $D_i\cap X=\{p_i\}$, for $i=r+3,\ldots,k$. Take $Y:=(\mathbb{P}^1)_1\cup_\lambda(\mathbb{P}^1)_2$ a union of two lines identified at λ. We construct a degree k admissible covering $f':X'\to Y$ as follows: take $f'_{|C}=f_1:C\to$

$(\mathbb{P}^1)_1$, $f'_{|E} = f_2 : E \to (\mathbb{P}^1)_2$ a map of degree $r+2$ sending the points $p_1, \ldots, p_{r+2}$ to λ, and finally $f'_{|D_i} : D_i \simeq (\mathbb{P}^1)_2$ isomorphisms sending p_i to λ. Clearly f' is an admissible covering, so X which is stably equivalent to X' is a k-gonal curve.

Let us consider now the space $\overline{\mathcal{H}}_{g+r+1,k}$ of Harris-Mumford admissible coverings of degree k (cf. [HM]) and denote by $\pi_1 : \overline{\mathcal{H}}_{g+r+1,k} \to \overline{\mathcal{M}}_{g+r+1}$ the natural projection which sends a covering to the stable model of its source. We assume for simplicity that $\mathrm{Aut}(C) = \{Id_C\}$ which implies that $\mathrm{Aut}(f') = \{Id_{X'}\}$, so $[f']$ is a smooth point of $\overline{\mathcal{H}}_{g+r+1,k}$. In the case when C has nontrivial automorphisms the argument carries through without change if we replace the space of admisible coverings with the space of twisted covers of Abramovich, Corti and Vistoli (cf. [ACV]).

We compute the differential of the map π_1 at $[f']$. We have $T_{[f']}(\overline{\mathcal{H}}_{g+r+1,k}) = \mathrm{Def}(X', f', Y) = \mathrm{Def}(X, f, Y)$, where $f = f'_{|X} : X \to Y$. The differential $(d\pi_1)_{[f']}$ is the forgetful map $\mathrm{Def}(X, f, Y) \to \mathrm{Def}(X)$ and from the sequence (2.1) we get that $\mathrm{Im}(d\pi_1)_{[f']} = u_1^{-1}(\mathrm{Im}\ u_2)$, where $u_1 : \mathrm{Def}(X) \to \mathrm{Ext}^1(f^*\Omega_Y, \mathcal{O}_X)$ and $u_2 : \mathrm{Def}(Y) \to \mathrm{Ext}^1(f^*\Omega_Y, \mathcal{O}_X)$ are the dual maps of $u_1^\vee : H^0(X, \omega_X \otimes f^*\Omega_Y) \to H^0(X, \omega_X \otimes \Omega_X)$ and $u_2^\vee : H^0(X, \omega_X \otimes f^*\Omega_Y) \to H^0(Y, \omega_Y \otimes \Omega_Y)$. Here $u_2^\vee$ is induced by the trace map $\mathrm{tr} : f_*\omega_X \to \omega_Y$. Starting with the exact sequence on X,

$$0 \longrightarrow \mathrm{Tors}(\omega_X \otimes \Omega_X) \longrightarrow \omega_X \otimes \Omega_X \longrightarrow \Omega_C^{\otimes 2}(\Delta) \oplus \Omega_E^{\otimes 2}(\Delta) \longrightarrow 0,$$

we can write the following commutative diagram of sequences

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^0(\mathrm{Tors}(\omega_X \otimes f^*\Omega_Y)) & \hookrightarrow & H^0(\omega_X \otimes f^*\Omega_Y) & \twoheadrightarrow & H^0(2K_C - R_1 + \Delta) \oplus H^0(2K_E - R_2 + \Delta) \\
\Big\downarrow{\scriptstyle (u_1^\vee)_{\mathrm{tors}}} & & \Big\downarrow{\scriptstyle u_1^\vee} & & \Big\downarrow \\
H^0(\mathrm{Tors}(\omega_X \otimes \Omega_X)) & \hookrightarrow & H^0(\omega_X \otimes \Omega_X) & \twoheadrightarrow & H^0(2K_C + \Delta) \oplus H^0(2K_E + \Delta)
\end{array}$$

where R_1 (resp. R_2) is the ramification divisor of the map f_1 (resp. f_2). Taking into account that $H^0(E, 2K_E - R_2 + \Delta) = 0$ and that $H^0(Y, \omega_Y \otimes \Omega_Y) = H^0(\mathrm{Tors}(\omega_Y \otimes \Omega_Y))$, we obtain that

$$\mathrm{Im}(d\pi_1)_{[f']} = \big(H^0(C, 2K_C - R_1 + \Delta) \oplus \mathrm{Ker}(u_2^\vee)_{\mathrm{tors}}\big)^\perp, \tag{6}$$

where $(u_2^\vee)_{\mathrm{tors}} : H^0(\mathrm{Tors}(\omega_X \otimes f^*\Omega_Y)) \to H^0(\mathrm{Tors}(\omega_Y \otimes \Omega_Y))$ is the restriction of $u_2^\vee$. The space $\mathrm{Ker}(u_2^\vee)_{\mathrm{tors}}$ is just a hyperplane in $H^0(\mathrm{Tors}(\omega_X \otimes f^*\Omega_Y)) \simeq \mathbb{C}^{r+2}$.

Intermezzo. If we also assume that $\rho(g, 1, k) < 0$ and that $[C]$ is a smooth point of $M^1_{g,k}$ (which happens precisely when $\mathrm{Aut}(C) = \{Id_C\}$, C has exactly one $\mathfrak{g}^1_k$ and $\dim|2\mathfrak{g}^1_k| = 2$), then we can prove that the locus $\overline{M}^1_{g+r+1,k}$ is smooth at $[X]$ as well. Indeed, since $\Delta \in C_{r+2}$ was chosen generically in a fibre of the $\mathfrak{g}^1_k$ on C, from Riemann-Roch we have that $h^0(C, 2K_C - R_1 + \Delta) = g - 2k + 3 + r = \mathrm{codim}(\overline{M}^1_{g+r+1,k}, \overline{M}_{g+r+1})$. The fibre over $[X]$ of the map $\pi_1 : \overline{\mathcal{H}}_{g+r+1,k} \to \overline{M}_{g+r+1}$ is identified with the space of degree $r+1$ maps $f_2 : E \to \mathbb{P}^1$ such that $f_2(p_1) = \ldots = f_2(p_{r+2}) = \lambda$, hence it is $r+1$ dimensional. We compute the tangent cone

$$TC_{[X]}(\overline{M}^1_{g+r+1,k}) = \bigcup\{\mathrm{Im}(d\pi_1)_z : z \in \pi_1^{-1}([X])\} = H^0(C, 2K_C - R_1 + \Delta)^{\perp},$$

which shows that $[X]$ is a smooth point of the locus $\overline{M}^1_{g+r+1,k}$.

We compute now the differential

$$(d\pi_2)_{[X]} : T_{[X]}(\mathrm{Hilb}_{d+r,g+r+1,r}) \to T_{[X]}(\overline{\mathcal{M}}_{g+r+1}),$$

which is the same thing as the differential at the point $[X \hookrightarrow \mathbb{P}^r]$ of the projection $\pi_2 : \overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r) \to \overline{\mathcal{M}}_{g+r+1}$. We start by noticing that X is smoothable in $\mathbb{P}^r$ and that $H^1(X, N_X) = 0$ (apply Proposition 2.1). We also have that X is embedded in $\mathbb{P}^r$ by a complete linear system, that is, $h^0(X, \mathcal{O}_X(1)) = r+1$. Indeed, on one hand, since X is nondegenerate, $h^0(X, \mathcal{O}_X(1)) \geq h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r(1)}) = r+1$; on the other hand from the sequence (2) we have that $h^0(X, \mathcal{O}_X(1)) \leq h^0(C, \mathcal{O}_C(1)) = r+1$.

If X is embedded in $\mathbb{P}^r$ by a complete linear system, we know (cf. Section 2) that

$$\mathrm{Im}(d\pi_2)_{[X]} = (\mathrm{Im}\mu_1(X))^{\perp},$$

where $\mu_1(X) : \mathrm{Ker}\mu_0(X) \to H^0(X, \omega_X \otimes \Omega_X)$ is the 'derivative' of the Petri map $\mu_0(X) : H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) \to H^0(X, \omega_X)$. We compute the kernel of $\mu_0(X)$ and show that $\mu_0(X)$ is surjective in a way that resembles the proof of Proposition 2.3 in [Se].

From the sequence (3) we obtain $H^0(X, \omega_X) = H^0(C, K_C + \Delta)$, while from (2) we have that $H^0(X, \mathcal{O}_X(1)) = H^0(E, \mathcal{O}_E(1))$ (keeping in mind that $H^0(C, \mathcal{O}_C(1)(-\Delta)) = 0$, as $p_1, \ldots, p_{r+2}$ are in general linear position). Finally, using (3) again, we have that $H^0(X, \omega_X(-1)) = H^0(C, K_C(-1) + \Delta)$. Therefore we can write the following commutative diagram:

$$\begin{array}{ccc}
H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1)) & \xrightarrow{\mu_0(C)} & H^0(C, K_C) \\
\downarrow & & \downarrow \\
H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1) + \Delta) & \longrightarrow & H^0(C, K_C + \Delta) \\
\downarrow = & & \downarrow = \\
H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) & \xrightarrow{\mu_0(X)} & H^0(X, \omega_X)\ .
\end{array}$$

It follows that $\mathrm{Ker}\mu_0(C) \subseteq \mathrm{Ker}\mu_0(X)$. By Corollary 1.6 from [CR], our assumptions ($\mu_0(C)$ surjective and $\mathrm{card}(\Delta) \geq 4$) enable us to conclude that $\mu_0(X)$ is surjective too. Then $\mathrm{Ker}\mu_0(C) = \mathrm{Ker}\mu_0(X)$ for dimension reasons, hence also $\mathrm{Im}\mu_1(X) = \mathrm{Im}\mu_1(C) \subseteq H^0(C, 2K_C) \subseteq H^0(X, \omega_X \otimes \Omega_X)$. We thus get that $\mathrm{Im}(d\pi_2)_{[X]} = (\mathrm{Im}\mu_1(X))^{\perp} = (\mathrm{Im}\mu_1(C))^{\perp}$.

The assumption that (C, f_1, f_2) satisfies (5) can be rewritten by passing to duals as

$$H^0(C, 2K_C - R_1)^{\perp} + (\mathrm{Im}\mu_1(C))^{\perp} = H^1(C, T_C) \Longleftrightarrow H^0(C, 2K_C - R_1) \cap \mathrm{Im}\mu_1(C) = 0.$$

Then it follows that $\mathrm{Im}\mu_1(X) \cap \big(H^0(C, 2K_C - R_1 + \Delta) \oplus \mathrm{Ker}((u_2^\vee)_{\mathrm{tors}})\big) = 0$, which is the same thing as

$$(d\pi_1)_{[f']}\ \big(T_{[f']}(\overline{\mathcal{H}}_{g+r+1,k})\big) + (d\pi_2)_{[X\hookrightarrow\mathbb{P}^r]}\ \big(T_{[X\hookrightarrow\mathbb{P}^r]}(\overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r))\big) = \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X). \tag{7}$$

This means that the images of $\overline{\mathcal{H}}_{g+r+1,k}$ under the map π_1 and of $\overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r)$ under the map π_2, meet transversally at the point $[X] \in \overline{\mathcal{M}}_{g+r+1}$.

Claim. The curve X can be smoothed in such a way that the $\mathfrak{g}^1_k$ and the very ample $\mathfrak{g}^r_{d+r}$ are preserved (while (7) is an open condition on $\overline{\mathcal{H}}_{g+r+1,k} \times \overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r)$).

Indeed, the tangent directions that fail to smooth at least one node of X are those in $\bigcup_{i=1}^{r+2} H^0(\mathrm{Tors}_{p_i}(\omega_X \otimes \Omega_X))^\perp$, whereas the tangent directions that preserve both the $\mathfrak{g}^1_k$ and the $\mathfrak{g}^r_{d+r}$ are those in

$$\big((\mathrm{Im}\mu_1(C) + H^0(C, 2K_C - R_1 + \Delta)) \oplus \mathrm{Ker}(u_2^\vee)_{\mathrm{tors}}\big)^\perp.$$

Since $H^0(\mathrm{Tors}_{p_i}(\omega_X \otimes \Omega_X)) \nsubseteq \mathrm{Ker}(u_2^\vee)_{\mathrm{tors}}$ for $i = 1, \ldots, r+2$, by moving in a suitable direction in the tangent space at $[f']$ of $\pi_1^{-1}\pi_2(\overline{\mathcal{M}}_{g+r+1}(\mathbb{P}^r, d+r))$, we finally obtain a smooth curve $Y \subseteq \mathbb{P}^r$ with $g(Y) = g+r+1, \deg(Y) = d+r$ and satisfying all the required properties. □

Using the previous result together with Proposition 1.1 we construct now regular components of $\mathcal{M}_g\big(\mathbb{P}^1 \times \mathbb{P}^r, (k,d)\big)$.

Theorem 1 *Let g, r, d and k be positive integers such that $r \geq 3$, $\rho(g,r,d) < 0$ and*

$$(2 - \rho(g,r,d))r + 2 \leq k \leq (g+2)/2.$$

Then there exists a regular component of the stack of maps $\mathcal{M}_g\big(\mathbb{P}^1 \times \mathbb{P}^r, (k,d)\big)$.

Proof. All integer solutions (g_0, d_0) of the equation $\rho(g_0, r, d_0) = \rho(g, r, d)$ with $g_0 \leq g$ and $d_0 \leq d$, are of the form $g_0 = g - a(r+1)$ and $d_0 = d - ar$ with $a \geq 0$. Using our numerical assumptions, by a routine check we find that there exists $a > 0$ such that $g_0 = g - a(r+1) > 0$, $d_0 = d - ar \geq r+1$, $k \geq g_0 + 1$ and

$$-\frac{g_0}{r} + \frac{r+1}{r} \leq \rho(g_0, r, d_0) < 0.$$

By Proposition 1.1 there exists a smooth curve $C_0 \subseteq \mathbb{P}^r$ of genus g_0 and degree d_0, with $H^1(C_0, N_{C_0/\mathbb{P}^r}) = 0$, $h^0(C_0, \mathcal{O}_{C_0}(1)) = r+1$ and $\mu_0(C_0)$ surjective. Moreover, since $k \geq g_0 + 1$, there exists an open dense subset $U \subseteq \mathrm{Pic}^k(C_0)$ such that for each $L_1 \in U$ there exists a pencil $l_1 = (L_1, V_1) \in G^1_k(C_0)$ with $V_1 \in \mathrm{Gr}(2, H^0(C_0, L))$, such that l_1 is simple and base point free (cf. [Fu, Proposition 8.1]).

We denote by $\pi_1 : \mathcal{M}_{g_0}(\mathbb{P}^1, k_0) \to \mathcal{M}_{g_0}$ the natural projection and by $f_1 : C \to \mathbb{P}^1$ the map corresponding to l_1. By Riemann-Roch we have $H^1(C_0, L_1^{\otimes 2}) = 0$, hence using

Section 2 $(d\pi_1)_{[f_1]} : T_{[f_1]}(\mathcal{M}_{g_0}(\mathbb{P}^1, k)) \to T_{[C_0]}(\mathcal{M}_{g_0})$ is surjective since $\mathrm{Coker}(d\pi_1)_{[f_1]} = H^1(C_0, f_1^* T_{\mathbb{P}^1}) = 0$. It follows that $(C_0, |\mathcal{O}_{C_0}(1)|, l_1)$ satisfies (5).

We claim that if $L \in U$ is general then $|\mathcal{O}_{C_0}(1) \otimes L^\vee| = \emptyset$. Suppose not, that is $\mathcal{O}_{C_0}(1) \otimes L^\vee \in W_{d_0-k}(C_0)$ for a general $L \in \mathrm{Pic}^k(C_0)$. This is possible only for $d_0 - k \geq g_0$, hence

$$r + 2 \leq k \leq d_0 - g_0 < r, \text{ (because } \rho(g_0, r, d_0) = \rho(g, r, d) < 0),$$

a contradiction. Thus $(C_0, |\mathcal{O}_{C_0}(1)|, l_1)$ satisfies all conditions required by Proposition 3.1 which we can now apply a times to get a smooth curve $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ of genus g and bidegree (k, d) such that $H^1(C, N_{C/\mathbb{P}^1 \times \mathbb{P}^r}) = 0$. The conclusion of Theorem 1 now follows. □

In the special case $\rho(g, r, d) = -1$ we can extend the range of possible g, r, d and k for which there is a regular component:

Theorem 2 *Let g, r, d, k be positive integers such that $r \geq 3$, $\rho(g, r, d) = -1$ and*

$$\frac{2r^2 + r + 1}{r - 1} \leq k \leq \frac{g + 2}{2} .$$

Then there exists a regular component of the stack of maps $\mathcal{M}_g\big(\mathbb{P}^1 \times \mathbb{P}^r, (k, d)\big)$.

Proof. We find a solution $\big(g_0 = g - a(r+1), d_0 = d - ar\big)$ of the equation $\rho(g_0, r, d_0) = \rho(g, r, d)$ with $a \in \mathbb{Z}_{\geq 0}$ such that $d_0 \geq k + r$ and $\rho(g_0, 1, k) \geq r - 1$. Our numerical assumptions ensure that such an $a \geq 0$ exists. Note that in this case $k \leq g_0 + 1$, so we are not in the situation covered by Theorem 1.

It also follows that $-\frac{g_0}{r} + \frac{r+1}{r} \leq -1 = \rho(g_0, r, d_0)$ and $d_0 \geq r + 1$, hence by Proposition 1.1 there exists an irreducible smooth open subset U of $\mathcal{M}_{g_0}(\mathbb{P}^r, d_0)$ of the expected dimension, such that all points of U correspond to embeddings of smooth curves $C \hookrightarrow \mathbb{P}^r$, with $h^1(C, N_C) = 0$, $h^0(C, \mathcal{O}_C(1)) = r + 1$ and $\mu_0(C)$ surjective.

Since we are in the case $\rho(g_0, r, d_0) = -1$, a combination of results by Eisenbud, Harris and Steffen gives that the Brill-Noether locus $M^r_{g_0,d_0}$ is an irreducible divisor in M_{g_0} (see [St, Theorem 0.2]). It follows that the natural projection $\pi_2 : U \to \mathcal{M}^r_{g_0,d_0}$ is dominant.

To apply Proposition 3.1 we now find a curve $[C_0] \in M^{r_0}_{g_0,d_0}$ having a complete base point free $\mathfrak{g}^1_k$ such that $2\mathfrak{g}^1_k$ is non-special. Then by semicontinuity we get that the general $[C] \in U$ also possesses a pencil $\mathfrak{g}^1_k$ with these properties. To find one particular such curve we proceed as follows: take C_0 a general $(r+1)$-gonal curve of genus g_0. These curves will have rather few moduli $(r + 1 < [(g+3)/2])$ but we still have that $[C_0] \in M^r_{g_0,d_0}$. Indeed, according to [CM] we can construct a $\mathfrak{g}^r_{d_0} = |\mathfrak{g}^1_{r+1} + F|$ on C_0, where F is an effective divisor on C_0 with $h^0(C_0, F) = 1$. Since $k \leq g_0$, using Corollary 2.2.3 from [CKM] we find that C_0 also carries a complete base point free $\mathfrak{g}^1_k$, not composed with the $\mathfrak{g}^1_{r+1}$ computing $\mathrm{gon}(C_0)$, and such that $2\mathfrak{g}^1_k$ is non-special. Since these are open conditions, they will hold generically along a component of $G^1_k(C_0)$. Applying semicontinuity, for a general element $[C] \in M^r_{g_0,d_0}$ (hence also for a general element $[C] \in U$), the variety $G^1_k(C)$ will contain a component A with general point $l \in A$ being complete, base point free and with $2l$ non-special.

We claim that for a general $l \in A$ we have that $|\mathcal{O}_C(1)|(-l) = \emptyset$. Suppose not. Then if we denote by $V^{r-1}_{d_0-k}(|\mathcal{O}_C(1)|)$ the variety of effective divisors of degree $d_0 - k$ on C imposing $\leq r-1$ conditions on $|\mathcal{O}_C(1)|$, we obtain

$$\dim V^{r-1}_{d_0-k}(|\mathcal{O}_C(1)|) \geq \dim A \geq \rho(g_0, 1, k) \geq r-1.$$

Therefore $C \subseteq \mathbb{P}^r$ has at least ∞^{r-1} (d_0-k)-secant $(r-2)$-planes, hence also at least ∞^{r-1} r-secant $(r-2)$-planes (because $d_0 - k \geq r$). This last statement contradicts the Uniform Position Theorem (see [ACGH, p. 112]), hence the general point $[C] \in U$ enjoys all properties required to make Proposition 3.1 work. □

Remark. From the proof of Theorem 2 the following question appears naturally: let us fix g, k such that $g/2 + 1 \leq k \leq g$. One knows (cf. [ACGH]) that if $l \in G^1_k(C)$ is a complete, base point free pencil then $\dim T_l(G^1_k(C)) = \rho(g,1,k) + h^1(C, 2l)$. Therefore if A is a component of $G^1_k(C)$ such that $\dim A = \rho(g,1,k)$ and the general $l \in A$ is base point free such that $2l$ is special, then A is nonreduced. What is then the dimension of the locus

$$V_{g,k} := \{[C] \in M_g : \text{ every component of } G^1_k(C) \text{ is nonreduced } \}?$$

A result of Coppens (cf. [Co]) says that for a curve C, if the scheme $W^1_k(C)$ is reduced and of dimension $\rho(g,1,k)$, then the scheme $W^1_{k+1}(C)$ is reduced too and of dimension $\rho(g,1,k+1)$. It would make then sense to determine $\dim(V_{g,k})$ when $\rho(g,1,k) \in \{0,1\}$ (depending on the parity of g). We suspect that $V_{g,k}$ depends on very few moduli and if g is suitably large we expect that $V_{g,k} = \emptyset$.

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