

Abelian Varieties, 2/14/2006 - Notes taken by Pippa Charters

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Let $X = V/\Lambda$ be a torus, and recall that we have the canonical identification

$$NS(X) = \{H : V \times V \rightarrow \mathbb{C} : H \text{ Hermitian and } E = \text{Im}(H)(\Lambda, \Lambda) \subseteq \mathbb{Z}\} = \text{Image of } c_1.$$

This set turns out to be in one-to-one correspondence with the set of \mathbb{R} -alternating forms

$$\{E : V \times V \rightarrow \mathbb{R} : E(\Lambda, \Lambda) \subseteq \mathbb{Z} \text{ and } E(ix, iy) = E(x, y)\}.$$

Also, let

$$\mathcal{P}(\Lambda) := \{(H, \chi) : H \in NS(X), \chi : \Lambda \rightarrow S^1 \text{ a semi-character}\}$$

where χ being a semi-character for H means that we have $\chi(\lambda + \lambda') = \chi(\lambda)\chi(\lambda')e^{\pi i E(\lambda, \lambda')}$. Recall too that we have defined a map $\mathfrak{A} : \mathcal{P}(\Lambda) \rightarrow \text{Pic}(X) \cong H^1(\Lambda, H^0(\mathcal{O}_V^*))$ which we claimed was a group isomorphism. Here, we take $\mathfrak{A}(H, \chi) := L(H, \chi) = [\{a_\lambda(z)\}] \in H^1(\Lambda, H^0(\mathcal{O}_V^*))$, where

$$a_\lambda(z) := \chi(\lambda)e^{\pi H(z, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)}.$$

We then get the following diagram:

$$\begin{array}{ccc} \mathcal{P}(\Lambda) & \longrightarrow & \text{Pic}(X) \\ & \searrow p & \downarrow c_1 \\ & & NS(X) \end{array}$$

where p is the projection map, $p(H, \chi) = H$.

Proposition. *This diagram is commutative, that is, $c_1(L(H, \chi)) = H$.*

Proof. Recall that if we write $L = \{e_\lambda(z)\} = \{e^{2\pi i f_\lambda(z)}\}$ with $f_\lambda(z) \in H^0(\mathcal{O}_V)$, then $c_1(L) \in \text{Alt}_{\mathbb{Z}}^2(\Lambda, \mathbb{Z})$ is given by

$$(\lambda, \mu) \mapsto f_\lambda(z) - f_\mu(z) - f_\lambda(z + \mu) + f_\mu(z + \lambda).$$

For $a_\lambda(z)$, we can then write this as $e^{2\pi i [g(\lambda) - \frac{i}{2} H(z, \lambda) - \frac{i}{4} H(\lambda, \lambda)]}$, where $g(\lambda)$ is a lift of $\chi(\lambda)$, and

everything in $[\dots]$ is $f_\lambda(z)$. This results in the first Chern class being

$$\begin{aligned}
& g(\lambda) - \frac{i}{2}H(z, \lambda) - \frac{i}{4}H(\lambda, \lambda) \\
& -g(\mu) + \frac{i}{2}H(z, \mu) + \frac{i}{4}H(\mu, \mu) \\
& -g(\lambda) + \frac{i}{2}H(z + \mu, \lambda) + \frac{i}{4}H(\lambda, \lambda) \\
& +g(\mu) - \frac{i}{2}H(z + \lambda, \mu) - \frac{i}{4}H(\mu, \mu) = \frac{i}{2}H(\mu, \lambda) - \frac{i}{2}H(\lambda, \mu) \\
& = -\frac{i}{2}[H(\lambda, \mu) - \overline{H(\lambda, \mu)}] \\
& = \Im(H(\lambda, \mu)) = E(\lambda, \mu)
\end{aligned}$$

□

Theorem (Appell-Humbert Theorem). *The map $\mathfrak{A} : \mathcal{P}(\Lambda) \rightarrow \text{Pic}(X)$ is an isomorphism of groups, and moreover we have an isomorphism of exact sequences:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \text{Hom}(\Lambda, S^1) & \longrightarrow & \mathcal{P}(\Lambda) & \longrightarrow & \text{NS}(X) & \longrightarrow & 1 \\
\downarrow \cong & & \downarrow \mathfrak{A} & & \downarrow \mathfrak{A} & & \downarrow = & & \downarrow \cong \\
1 & \longrightarrow & \text{Pic}^0(X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & \text{NS}(X) & \longrightarrow & 1
\end{array}$$

where $\text{Pic}^0(X)$ is the group of topologically trivial line bundles on the torus X , i.e.,

$$\text{Pic}^0(X) = \{L \in \text{Pic}(X) : c_1(L) = 0\}.$$

[Note that $\text{Pic}^0(X) \cong \text{Hom}(\Lambda, S^1) \cong (S^1)^{2g}$, so $\text{Pic}^0(X)$ is a topological torus].

Proof. By the snake lemma, it's enough to show that $\mathfrak{A} : \text{Hom}(\Lambda, S^1) \rightarrow \text{Pic}^0(X)$ is an isomorphism.

\mathfrak{A} injective: Recall that $a_\lambda(z) = \chi(\lambda)$. Let $\mathfrak{A}(\chi) = 0 \in \text{Pic}^0(X)$. Then the line bundle $\{\chi(\lambda)\}_\lambda$ is trivial. This happens if and only if there exists a function $f \in H^0(\mathcal{O}_V^*)$ such that $\chi(\lambda) = \frac{f(z+\lambda)}{f(z)}$. Since $|\chi(\lambda)| = 1$, it follows that $f(z)$ is bounded. By Liouville's theorem, f is a constant and $\chi(\lambda) = 1$ as desired.

\mathfrak{A} surjective: First, claim that the map $H^1(X, \mathbb{C}) \rightarrow H^1(\mathcal{O}_X)$ in the following commutative diagram is surjective.

$$\begin{array}{ccccccc}
& & H^1(X, \mathbb{C}) & & & & \\
& & \downarrow \text{claim} & \searrow \epsilon & & & \\
\cdots & \longrightarrow & H^1(\mathcal{O}_X) & \xrightarrow{e^{2\pi i}} & H^1(\mathcal{O}_X^*) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) \longrightarrow \cdots
\end{array}$$

If this map is indeed surjective as claimed, then $\text{Pic}^0(X) = \ker(c_1) = \text{Im}(\epsilon)$. To see why this map is surjective, note that we in fact have the commutative diagram:

$$\begin{array}{ccc}
H^1(X, \mathbb{C}) & \longrightarrow & H^1(\mathcal{O}_X) \\
\parallel & & \nearrow \text{projection} \\
H^{0,1}(X) \oplus H^{1,0}(X) & & \\
\parallel & & \\
H^1(\mathcal{O}_X) \oplus H^0(\Omega'_X) & &
\end{array}$$

Since the map between the resolution of the constant sheaf \mathbb{C} in the deRham cohomology and the resolution of the structural sheaf \mathcal{O}_X in Dolbeault cohomology is given by projection, it follows that the induced cohomology map is projection as well. Surjectivity now follows.

Now consider a constant function $\mathbb{C} \rightarrow \mathcal{O}_X$ given by $u \mapsto e^{2\pi i u}$. We can then look at the commutative diagram

$$\begin{array}{ccc} H^1(X, \mathbb{C}) & \xleftarrow{\cong} & H^1(\Lambda, \mathbb{C}) \\ \epsilon \downarrow & & \downarrow \epsilon \\ H^1(\mathcal{O}_X^*) & \xleftarrow{\cong} & H^1(\Lambda, H^0(\mathcal{O}_V^*)) \end{array}$$

Note that Λ acts trivially on $H^1(\Lambda, \mathbb{C})$ since translation fixes a constant function: $f(x + \lambda) = f(x)$ for any λ . Λ acts by translation on $H^1(\Lambda, H^0(\mathcal{O}_V^*))$. It follows that any element L in the image of ϵ can be given by constant factors of automorphy. Thus there exists an automorphism $e : \Lambda \rightarrow \mathbb{C}^*$ such that $L = \{e_\lambda\}$ where $e_\lambda = e(\lambda)$. We must show that there exists a homomorphism $\chi : \Lambda \rightarrow S^1$ and some $f \in H^0(\mathcal{O}_V^*)$ such that $\chi(\lambda) = \frac{f(z+\lambda)}{f(z)}e(\lambda)$. We write $e(\lambda) = e^{2\pi i g(\lambda)}$, where $g(\lambda + \mu) \equiv g(\lambda) + g(\mu) \pmod{\mathbb{Z}}$. Note that $\text{Im}(g) : \Lambda \rightarrow \mathbb{R}$ is a homomorphism, which we extend by scalars to get $m : V \rightarrow \mathbb{R}$ and $l : V \rightarrow \mathbb{C}$ where we define

$$l(z) := m(iz) + im(z).$$

We then define our $\chi : \Lambda \rightarrow S^1$ by $\chi(\lambda) = e(\lambda) \frac{e^{2\pi i l(z)}}{e^{2\pi i l(z+\lambda)}}$. Then χ is a semicharacter, equivalent to $\{e(\lambda)\}_{\lambda \in \Lambda}$. It follows that \mathfrak{A} is surjective. \square

1 Applications of Appell Humbert Theorem

Proposition. *Let $L = L(H, \chi) \in \text{Pic}(X)$, $x = \pi(v) \in X$ where $\pi : V \rightarrow X$ is the universal cover of X . Then if t_x is translation by $x \in X$, the pullback of L is given by*

$$t_x^* L = L(H, \chi e^{2\pi i E(v, \cdot)}).$$

Proof. $c_1(t_x^* L) = c_1(L)$. Say $L = L(H, \chi)$. Then $\{a_\lambda(z)\}$ is a factor of automorphy for L . We then have the commutative diagram

$$\begin{array}{ccc} L(\pi(z)) & \xrightarrow{u_z} & \mathbb{C} \\ \parallel & & \downarrow a_\lambda(z) \\ L(\pi(z + \lambda)) & \xrightarrow{u_{z+\lambda}} & \mathbb{C} \end{array}$$

Then $t_x^*(L(z)) = L(x+z)$ therefore $\{a_\lambda(z+v)\}_{\lambda \in \Lambda}$ is a factor of automorphy for the translate $t_x^*(L)$, where v is a lift of x , but not necessarily the canonical one given by the Appel-Humbert theorem. Define

$$\tilde{a}_\lambda(z) = \chi(\lambda) e^{2\pi i E(v, \lambda)} e^{\pi H(z, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)}$$

We then recall that $H(v, \lambda) = E(iv, \lambda) + iE(v, \lambda)$ and check that

$$\begin{aligned} \frac{\tilde{a}_\lambda(z)}{a_\lambda(z+v)} &= e^{2\pi i E(v, \lambda) - \pi H(v, \lambda)} \\ &= e^{-\pi E(iv, \lambda) + \pi i E(v, \lambda)} \\ &= e^{-\pi H(\lambda, v)} \end{aligned}$$

Now set $f(z) = e^{-\pi H(z,v)}$, and note that

$$\frac{f(z+\lambda)}{f(z)} = \frac{\tilde{a}_\lambda(z)}{a_\lambda(z+v)}$$

which shows that the factors of automorphy $\{\tilde{a}_\lambda\}$ and $\{a_\lambda(z+v)\}$ are equivalent. \square

2 Theorem of the Square

Theorem. *Let $x, y \in X$ and $L \in \text{Pic}(X)$. Then $t_x^*(L) \otimes t_y^*(L) = t_{x+y}^*(L) \otimes L$.*

Proof. Let $L = L(H, \chi)$, $x = \pi(v)$ and $y = \pi(w)$ where $v, w \in V$ the universal covering space of X . Then $t_x^*(L(H, \chi)) \otimes t_y^*(L(H, \chi)) = L(H, \chi e^{2\pi i E(v, \cdot)}) \otimes L(H, \chi e^{2\pi i E(w, \cdot)}) = L(2H, \chi^2 e^{2\pi i E(v+w, \cdot)}) = L(H, \chi) \otimes t_{x+y}^* L(H, \chi)$. \square

Example. *Let $L = \mathcal{O}_X(D)$, where $D \subseteq X$ is a divisor. Then the Theorem of the square can be rewritten in divisor language as*

$$(D+x) + (D+y) \equiv D + (D+x+y),$$

where \equiv denotes linear equivalence of divisors.

Proposition. *Let $X = V/\Lambda$, $Y = V'/\Lambda'$ and $f : X \rightarrow Y$ a homomorphism of tori. Let $\rho_a(f) = F : V \rightarrow V'$ be the analytic representation of f with corresponding restriction map $F_\Lambda : \Lambda \rightarrow \Lambda'$. Then*

$$f^* L(H, \chi) = L(F^* H, F_\Lambda^* \chi).$$

I.e., we have the commutative diagram:

$$\begin{array}{ccc} V' \times V' & \xrightarrow{H} & \mathbb{C} \\ F \times F \uparrow & \nearrow F^* H & \\ V \times V & & \end{array}$$

Proof. $f^*(H, \chi)(x) = L(H, \chi)(f(x))$. The pullback of $\{a_\lambda(z)\}$ (the canonical factors of automorphy for L) will be a factor of automorphy for $f^*(L(H, \chi))$, which is actually the canonical factor of automorphy. \square

Corollary. *If $[n]_X : X \rightarrow X$ is multiplication by n , then*

$$[n]_x^*(\mathcal{L}) = \mathcal{L}^{\frac{n(n+1)}{2}} \otimes [-1]_x^*(\mathcal{L}^{\frac{n(n-1)}{2}})$$

.

Definition. *We say that a line bundle $L \in \text{Pic}(X)$ is symmetric if $[-1]_X^*(L) = L$.*

For symmetric bundles then we have $[n]_X^*(\mathcal{L}) = \mathcal{L}^{n^2}$. We have the following characterization of symmetric bundles in terms of semi-characters:

Remark. $L = L(H, \chi)$ is symmetric $\iff \chi^2 = 1$ or $\chi(\Lambda) \subseteq \{\pm 1\}$.

Proof of Corollary. On the left hand side, we have $[n]_X^* \mathcal{L}(H, \chi) = \mathcal{L}(n^2 H, \chi(n\lambda))$. On the right hand side, we have $L^{\frac{n(n+1)}{2}} \otimes [-1]_X^* L^{\frac{n(n-1)}{2}} = L(\frac{n(n+1)}{2} H, \chi^{\frac{n(n+1)}{2}}) \otimes L(\frac{n(n-1)}{2} H, (\chi^{-1})^{\frac{n(n-1)}{2}}) = L(n^2 H, \chi(n\lambda))$. \square