ABELIAN VARIETIES, APRIL 27TH 2006 - NOTES TAKEN BY RICARDO CONCEICAO

Recall that we have an identification between the theta divisor of the Jacobian and the locus $W_{g-1}(C)$ of line bundles on C of degree $g-1$ with at least one non-zero section.

Theorem 0.1. *Using the identification*

$$\Theta = W_{g-1}(C) = \{L \in \mathrm{Pic}^{g-1}(C) : h^0(L) \geq 1\},$$

we have that

$$\mathrm{mult}_L(\Theta) = h^0(L).$$

So far we have proved that $\mathrm{mult}_L(\Theta) \geq h^0(L)$.

Now we'll prove that $\mathrm{mult}_L(\Theta) \leq r+1$, suuming that $h^0(L) = r+1$. For that, we'll find a morphism $\alpha : C_{r+1} \longrightarrow \mathrm{Pic}^{g-1}(C)$ such that $\alpha^*(\Theta)$ is a union of $r+1$ smooth divisors. Since the union of $r+1$ smooth divisors has multiplicity less or equal than $r+1$, this would show that

$$\mathrm{mult}_L(\Theta) < \mathrm{mult}_{\alpha(L)} \alpha^*(\Theta) \leq r+1$$

which is the desired inequality.

Pick general points $p_1, ..., p_{r+1} \in C$ such that the divisor $E = p_1 + ... + p_{r+1}$ satisfies

(1) $h^0(L-E) = 0$
(2) $h^0(L+E) = h^0(L)$

It's not hard to see that one can choose E such that (1) is satisfied. For (2), we should notice that Riemann-Roch give us:

$$h^0(L+E) - h^0(K_C - L - E) = g - 1 + r + 1 + 1 - g = r + 1$$
$$h^0(L) - h^0(K_C - L) = g - 1 + 1 - g = 0$$

which amounts to say that

$$h^0(L) = h^0(L+E) \Longleftrightarrow h^0(K_C - L - E) = h^0(K_C - L) - (r+1)$$

which is easily seen to be verified for a general divisor E of degree r.

Now consider $F \in C_r$. Define $\alpha(F) = L \otimes \mathcal{O}_C(E-F)$. Then $\alpha^*(\Theta) = \{F \in C_r : h^0(L \otimes \mathcal{O}_C(E-F)) \geq 1\}$ or equivalently $\alpha^*(\Theta) = \{F \in C_r : -F + E + L \equiv A, A \geq 0\}$. By (2) we can find an effective $D \in L$ with $D \equiv L$ such that $D + E = F + A$. Now consider:

Case 1: Exist an $1 \leq i \leq r+1$ such that $p_i \in F$. In that case $F = p_i + F'$ where $F' \in C_r - 1$ and so $X_i = \{F : p_i \in F\} = p_i + C_{r-1}$ is smooth.

Case 2: $p_i \notin F, \forall i \in \{1, ..., r\}$. This implies that $F \subset D$, hence $h^0(L-F) \geq 0$. So $Y = \{F \in C_r : h^0(L-F) \geq 0\}$ is smooth.

Therefore $\alpha^*(\Theta) = \sum_{i=1}^{r+1} X_i + Y$. Notice that the result will follow if $\cap_{i=1}^{r+1} X_i \cap Y = \emptyset$. So assume $F \in \cap_{i=1}^{r+1} X_i \cap Y$ then $F = p_1 + ... + p_{r+1} + \tilde{F} = E + \tilde{F}$ with $\tilde{F} \geq 0$. Hence $h^0(L - F) \geq 1$ if and only if $h^0(L - E - \tilde{F}) \geq 1$, contradicting (1).

1. The singular locus of the theta divisor

In this section we'll study the singular loci of the Θ divisor. Let

$$\mathrm{Sing}(\Theta) = \{L \in \mathrm{Pic}^{g-1}(C) : h^0(L) \geq 2\} = W^1_{g-1}(C)$$

Now consider $C^1_{g-1} = \{D \in C_{g-1} : h^0(D) \geq 2\}$. Then we have

$$C^1_{g-1} \xrightarrow{\phi} W^1_{g-1}(C)$$

where ϕ is the Abel-Jacobi map, whose generic fibers are rational lines. By Riemann-Roch, we have $h^0(D) - h^o(K_C - D) = 0$, so that $h^0(K_C - D) \geq 2$. Let's consider the short exact sequence

$$0 \longrightarrow K_C(-D) \longrightarrow K_C \longrightarrow K_C|_D \longrightarrow 0,$$

which implies a long exact sequence in cohomology

$$0 \longrightarrow H^0(K_C - D) \longrightarrow H^0(K_C) \xrightarrow{\phi(D)} H^0(K_C|_D) \longrightarrow \cdots$$

This shows that $C^1_{g-1} = \{D \in C_{g-1} : \dim \mathrm{Ker}\, \phi(D) \geq 2\}$

Over C^1_{g-1} we can consider 2 vector bundles:

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\phi} & \mathbb{F} \\ & \searrow \quad \swarrow & \\ & C^1_{g-1} & \end{array}$$

Define $\mathbb{E}$ by $\mathbb{E}(D) = H^0(K_C)$, that is, $\mathbb{E} = H^0(K_C) \otimes \mathcal{O}_{C^1_{g-1}}$, the trivial bundle of rank g. Let $\mathbb{F}(D) = H^0(K_C|_D)$ and for this bundle we have $\mathrm{rank}(\mathbb{F}) = \deg(D) = g - 1$. Also if we denote the fiber of the canonical bundle at p_i by $K_C(p_i)$, then

$$\mathbb{E}(D) = K_C(p_1) \oplus \ldots \oplus K_C(p_{g-1}) \simeq \mathbb{C}^{g-1}$$

So

$$\begin{aligned} C^1_{g-1} &= \{D \in C_{g-1} : \mathrm{rank}\, \phi(D) \leq g - 2\} \\ &= \{D \in C_{g-1} : \text{rank of the matrix giving } \phi \text{ in local coordinates is } \leq g - 2\} \end{aligned}$$

It is a general linear algebra fact that the space

$$\{A \in M_{m,n}(\mathbb{C}) : \mathrm{rank}(A) \leq k\}$$

has dimension $mn-(m-k)(n-k)$. For $m=g$, $n=g-1$ and $k=g-2$, using the theorem on the dimension of the fibres of a morphism, this implies that $\operatorname{codim}(C^1_{g-1}, C_{g-1}) \leq 2$, hence $\dim C^1_{g-1} \geq g-3$ and $\dim \operatorname{Sing}(\Theta) \geq g-4$.

Remark: Varieties, like C^1_{g-1}, that are given by linear conditions are called *determinantal varieties*. The locus $W^1_{g-1}(C)$ is an example of a Brill-Noether locus associated to the curve C.

Denote by $\mathcal{A}_g = \{(X,\Theta)$: principally polarized Abelian Varieties$\}$ and $\mathcal{M}_g = \{[C] : C$ smooth curve of genus $g\}$. There's a well-defined map

$$t : \mathcal{M}_g \longrightarrow \mathcal{A}_g$$

given by $t(C) = (\operatorname{Jac}(C), \Theta_C)$. Define a stratification of the moduli space

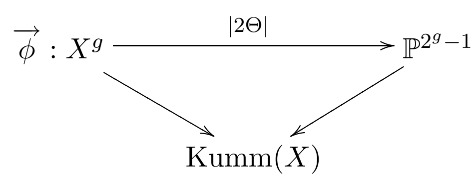
$$N_k = \{(X,\Theta) \in \mathcal{A}_g : \dim \operatorname{Sing}(\Theta) \geq k\}.$$

Thus we obtain

$$\mathcal{A}_g \supsetneq N_0 \supseteq \ldots \supseteq N_{g-4} \supseteq \ldots \supseteq N_{g-1}$$

We just proved that $\mathcal{M} \subseteq N_{g-4}$. On the other hand, for a generic $(X,\Theta) \in \mathcal{A}_g$ we have that Θ smooth (to be proved later). A natural question to consider is if the condition $\dim \operatorname{Sing}(\Theta) \geq g-4$ can be used to characterize $\mathcal{M}_g$ inside $\mathcal{A}_g$. Unfortunately this is not possible, but one can prove that $\mathcal{M}_g$ is a irreducible component of $N_g - 4$.

How could someone characterize Jacobians geometrically? One way would be to look at the linear system $|2\Theta|$ and the map given by it. Since $\overrightarrow{\phi}(x) = \overrightarrow{\phi}(-x)$, if we denote by $\operatorname{Kumm}(X) = \dfrac{X}{x \sim -x}$ the Kummer variety of X and we obtain the following commutative diagram



It happens that using this one can show that Jacobians have a 4 dimensional family of trisecants lines in the $|2\Theta|$ embedding, if $g \geq 3$. To prove this let's prove a initial result called *Riemann's Adition Formula*.

Recall that given $(X = V/\Lambda, \Theta)$ one can define the following decompositions and objects:

$\Lambda = \Lambda_1 \oplus \Lambda_2$, $k(L) = k(L)_1 \oplus k(L)_2$, $L = \mathcal{O}_X(\Theta)$, $c_1(L) = H \in \operatorname{NS}(X)$ with type of H equals to $(d_1, \ldots, d_g)$ and $\{\vartheta_w(z)\}_{w \in k(L)_1}$ a basis for $H^0(X, L)$.

Suppose X has a principal polarization H, so that $H^0(X, \mathcal{O}_X(\Theta))$ is generated by one single theta function $\vartheta(z)$ and $2H$ has type $(2, \ldots, 2)$. Let ψ be the theta function associated to $|2\Theta|$. Then the functions $\psi_w(z) = \dfrac{1}{a_w(z)} \psi_\psi(z+w)$ satisfy

$$H^0(X, \mathcal{O}_X(2\Theta)) = \mathcal{C} < \psi_w(z) >_{w \in k(2H)_1}$$

Theorem 1.1.

$$\vartheta(z+a)\vartheta(z-a) = \sum_{w \in k(2H_1} \psi_w(a)\psi_w(z)$$

Proof: Fixing $a \in V$ as $z \in V$ varies we have $\sum_{w \in k(2H_1} \psi_w(a)\psi_w(z) \in H^0(X, \mathcal{O}_X(2\Theta))$. So $|2\Theta| \supseteq (\Theta + a) + (\Theta - a))$. Recall that Θ is the zeros loci of ϑ, hence $\Theta + a = Z(\vartheta(z-a) = 0)$ and $\Theta - a = Z(\vartheta(z+a) = 0)$, therefore the left-hand side in the theorem claim gives an equation for $(\Theta + a) + (\Theta - a)$, as we wanted to prove. □