

1 Thursday 1/26

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1.1 Complex Geometry

Last time we talked about the cohomology of complex tori $X = V/\Lambda$, where $V = \mathbb{C}^g$ and $\Lambda \subset V$ is a lattice. We showed that $H^1(X, \mathbb{Z}) = \Lambda^\vee = \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$, and then we showed that

$$H^n(X, \mathbb{Z}) = \bigwedge^n (\Lambda^\vee) = \text{Alt}_{\mathbb{Z}}^n(\Lambda, \mathbb{Z}).$$

The elements of $\text{Alt}_{\mathbb{Z}}^n(\Lambda, \mathbb{Z})$ are just alternating forms $\omega : \Lambda \times \cdots \times \Lambda \rightarrow \mathbb{Z}$. By extension of scalars we showed that

$$H^n(X, \mathbb{C}) = \text{Alt}_{\mathbb{R}}^n(V, \mathbb{C}).$$

Recall that one takes the wedge product of the 1-forms ℓ_1, \dots, ℓ_n on Λ by setting

$$\ell_1 \wedge \cdots \wedge \ell_n(v_1, \dots, v_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \ell_1(v_{\sigma(1)}) \cdots \ell_n(v_{\sigma(n)}),$$

which is the “ $n \times n$ determinant.” We also mentioned that $H^n(X, \mathbb{C})$ is (isomorphic to) the de Rham cohomology so

$$\text{Alt}_{\mathbb{R}}^n(V, \mathbb{C}) = \{\text{translation-invariant } n\text{-forms on } X\}.$$

(On a sidenote, we emphasize that the subscript \mathbb{R} on Alt does in fact indicate that our forms are \mathbb{R} -linear, not \mathbb{C} -linear, and in fact $H^1(X, \mathbb{C}) \simeq \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$.)

In any book on abelian varieties, it is often assumed that the reader is familiar with Hodge theory, or if the book is written from a more algebrogeometric perspective, with scheme theory. We will avoid these assumptions in this course and instead review some complex geometry today.

Let X be a complex manifold of dimension g . For $p \in X$, we have a number of different ways to the tangent space at p . The tangent space of the underlying real manifold X/\mathbb{R} is defined as

$$T_{X, \mathbb{R}}(p) = \{\delta : \mathcal{C}_{X, \mathbb{R}}^\infty(p) \rightarrow \mathbb{R} \text{ derivations}\}$$

where $\mathcal{C}_{X, \mathbb{R}}^\infty(p)$ denotes the stalk of germs of C^∞ functions on X at p . We will use $\mathcal{O}_{X, p}$ to denote the germs of holomorphic functions on X around p . If we let $z_j = x_j + iy_j$ be a set of coordinates, then

$$T_{X, \mathbb{R}}(p) = \mathbb{R} \left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \right\}$$

and the *complexified tangent space* is obtained by extending the scalars:

$$\begin{aligned} T_{X, \mathbb{C}}(p) &= T_{X, \mathbb{R}}(p) \otimes_{\mathbb{R}} \mathbb{C} = \{\mathbb{C}\text{-linear derivations } \delta : \mathcal{C}_{X, \mathbb{C}}^\infty(p) \rightarrow \mathbb{C}\} \\ &= \mathbb{C} \left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \right\} \end{aligned}$$

Define

$$\begin{aligned}\frac{\partial}{\partial z_j} &= \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \\ \frac{\partial}{\partial \bar{z}_j} &= \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right),\end{aligned}$$

then we can write our complexified tangent space

$$T_{X,\mathbb{C}}(p) = \mathbb{C} \left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \right\} = \mathbb{C} \left\{ \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \right\}.$$

Now of course the above construction works for any real manifold for even dimension, since we haven't made use of holomorphicity yet.

Define a splitting of the complexified tangent space into two pieces:

$$T_{X,\mathbb{C}}(p) = T'_X(p) \oplus T''_X(p)$$

where $T_X(p) = T'_X(p)$ is the *holomorphic tangent space* and $T''_X(p)$ is the *antiholomorphic tangent space*, given by

$$T'_X(p) = \mathbb{C} \left\{ \frac{\partial}{\partial z_j} \right\} = \{ \delta \in T_{X,\mathbb{C}}(p) : \delta \text{ vanishes on antiholomorphic germs} \}$$

$$T''_X(p) = \mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}_j} \right\} = \{ \delta \in T_{X,\mathbb{C}}(p) : \delta \text{ vanishes on holomorphic germs} \}$$

What hides behind this equality is that, as one might recall, $f \in \mathcal{C}_{X,\mathbb{C}}^\infty(p)$ is holomorphic iff $\partial f / \partial \bar{z}_j = 0 \forall j$, and similarly f is antiholomorphic iff $\partial f / \partial z_j = 0 \forall j$.

Everything we have said is functorial, so if $f : X \rightarrow Y$ is a C^∞ map then we have $df_p : T_{X,\mathbb{C}}(p) \rightarrow T_{Y,\mathbb{C}}(f(p))$. When is df_p compatible with our decomposition of $T_{X,\mathbb{C}}$ into holomorphic and antiholomorphic parts? It turns out that f is *holomorphic* if $df_p(T'_X(p)) \subseteq T'_Y(f(p))$ for all $p \in X$ (note that this implies that $df_p(T''_X(p)) \subseteq T''_Y(f(p))$, checking one condition is enough).

1.2 Forms and Cohomology

We now wish to discuss forms and cohomology. We have a splitting $T_{X,\mathbb{C}}(p) = T'_X(p) \oplus T''_X(p)$. By dualizing this we obtain

$$T_{X,\mathbb{C}}^\vee(p) = T'_X(p)^\vee \oplus T''_X(p)^\vee$$

and then we take wedge products:

$$\bigwedge^n T_{X,\mathbb{C}}^\vee(p) = \bigoplus_{\substack{a+b=n \\ a,b \geq 0}} \bigwedge^a T'_X(p)^\vee \otimes \bigwedge^b T''_X(p)^\vee.$$

Let $\mathcal{A}^n(X)$ be the space of C^∞ complex n -forms on X . Then given $\omega \in \mathcal{A}^n(X)$ at each $p \in X$ we have $\omega(p) : T_{X,\mathbb{C}}(p) \times \cdots \times T_{X,\mathbb{C}}(p) \rightarrow \mathbb{C}$ and we require that the $\omega(p)$ vary in a

continuous manner with p . In some neighborhood of each point p , the coefficients of ω are C^∞ functions.

So we have this rather large space of differentiable n -forms. Now the cotangent bundle splits, so we can talk about (a, b) -forms.¹ We note the space of (a, b) forms by

$$\mathcal{A}^{a,b}(X) = \{\omega \in \mathcal{A}^n(X) : \omega \in \bigwedge^a T'_X(p)^\vee \otimes \bigwedge^b T''_X(p)^\vee \ \forall p \in X\}$$

(of course throughout we have as convention that $a + b = n$ and $a, b \geq 0$). Any n -form can be written as the sum of its (a, b) parts, that is,

$$\mathcal{A}^n(X) = \bigoplus_{a+b=n} \mathcal{A}^{a,b}(X).$$

If we have coordinates z_1, \dots, z_g , then the cotangent space is generated by dx_j, dy_j , the dual basis to $\partial/\partial x_j, \partial/\partial y_j$, or we can consider the dz_j dual to $\partial/\partial z_j$, and we have the relations between the bases

$$\begin{aligned} dz_j &= dx_j + i dy_j \\ d\bar{z}_j &= dx_j - i dy_j. \end{aligned}$$

So $T'_X(p)$ is the complex vector space spanned by the dz_j and analogously for $T''_X(p)$ by the $d\bar{z}_j$; we write this as

$$\begin{aligned} T'_X(p) &= \mathbb{C}\{dz_j\}_{j=1}^g \\ T''_X(p) &= \mathbb{C}\{d\bar{z}_j\}_{j=1}^g. \end{aligned}$$

So what is the basis for $\mathcal{A}^{a,b}(X)$? We denote the basis by

$$\mathcal{A}^{a,b}(X) = \mathbb{C}\{dz_I \wedge d\bar{z}_J\}_{\substack{\#I=a \\ \#J=b}}$$

where by convention $I = \{i_1, \dots, i_a\}$ and $dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_a}$ and similarly for J and $d\bar{z}_J$.

So what do our (a, b) -forms look like? Given $\omega \in \mathcal{A}^{a,b}(X)$, we can write

$$\omega = \sum_{\substack{\#I=a \\ \#J=b}} f_{IJ} dz_I \wedge d\bar{z}_J$$

where the f_{IJ} are C^∞ functions.

Example 1. Suppose $\dim X = 1$, and our cotangent space is spanned by $dz, d\bar{z}$. Let's determine $\mathcal{A}^2(X) = \mathcal{A}^{2,0}(X) \oplus \mathcal{A}^{1,1}(X) \oplus \mathcal{A}^{0,2}(X)$. First, $\mathcal{A}^{0,2}(X)$ consists of forms $f d\bar{z} \wedge d\bar{z} = 0$, so $\mathcal{A}^{0,2}(X) = 0$. Similarly, $\mathcal{A}^{2,0}(X) = 0$ for dimension reasons as well, and so $\mathcal{A}^2(X) = \mathcal{A}^{1,1}(X) = \{\omega = f(z) dz \wedge d\bar{z} : f \in C^\infty\}$.

¹Note that these are usually referred to as (p, q) -forms, however we are already using p for a point in X and wish to avoid confusion.

Now let's move to a discussion of cohomology. Recall that we have a map given by exterior differentiation $d : \mathcal{A}^n(X) \rightarrow \mathcal{A}^{n+1}(X)$. Now, $\mathcal{A}^n(X)$ splits, so it is natural to ask what happens to $\mathcal{A}^{a,b}(X)$ under d . Suppose $\omega \in \mathcal{A}^{a,b}(X)$, and write

$$\omega = \sum_{\substack{\#I=a \\ \#J=b}} f_{IJ} dz_I \wedge d\bar{z}_J$$

then

$$d\omega = \sum_{\substack{\#I=a \\ \#J=b}} \left[\left(\sum_{i=1}^g \frac{\partial f_{IJ}}{\partial z_i} dz_i \right) \wedge dz_I \wedge d\bar{z}_J + \left(\sum_{j=1}^g \frac{\partial f_{IJ}}{\partial \bar{z}_j} d\bar{z}_j \right) \wedge dz_I \wedge d\bar{z}_J \right]$$

so in fact, $d\omega \in \mathcal{A}^{a+1,b}(X) \oplus \mathcal{A}^{a,b+1}(X)$. Let us write then

$$d\omega = \partial\omega + \bar{\partial}\omega$$

where $\partial\omega \in \mathcal{A}^{a+1,b}(X)$ and $\bar{\partial}\omega \in \mathcal{A}^{a,b+1}(X)$. Now clearly we have $\partial : \mathcal{A}^{a,b}(X) \rightarrow \mathcal{A}^{a+1,b}(X)$ by differentiation with respect to the holomorphic variables, and $\bar{\partial} : \mathcal{A}^{a,b}(X) \rightarrow \mathcal{A}^{a,b+1}(X)$ by differentiation with respect to the antiholomorphic variables.

Now we can define the Dolbeault cohomology. First, note that we have an exact sequence

$$\dots \longrightarrow \mathcal{A}^{a,b-1}(X) \xrightarrow{\bar{\partial}_{a,b-1}} \mathcal{A}^{a,b}(X) \xrightarrow{\bar{\partial}_{a,b}} \mathcal{A}^{a,b+1}(X) \longrightarrow \dots$$

Exercise 1.1. (Easy) Check that $\bar{\partial}_{a,b} \circ \bar{\partial}_{a,b-1} = 0$.

Definition 1. The (a,b) -Dolbeault cohomology group $H^{a,b}(X)$ is defined as

$$H^{a,b}(X) = \frac{\ker \bar{\partial}_{a,b}}{\text{im } \bar{\partial}_{a,b-1}}.$$

This has all been a lot of machinery, but what is important to remember is that we split the exterior differentiation operator and used this to define the Dolbeault cohomology group. This is a “good” cohomology in the sense that the cohomology of affine space is 0, like the Poincaré lemma proved for the de Rham cohomology of \mathbb{R}^n . We want contractible spaces to have trivial cohomology.

Let us note here that if $\omega \in \mathcal{A}^{p,0}(X)$, then $\omega = \sum_{\#I=p} f_I dz_I$ where the f_I are smooth but not necessarily holomorphic. If we require the f_I to be holomorphic, we obtain the (sub)space (also a sheaf) of holomorphic p -forms

$$\Omega^p(X) = \{\omega \in \mathcal{A}^{p,0}(X) \text{ s.t. } f_I \text{ holomorphic}\}.$$

In fact, the condition that the f_I be holomorphic is equivalent to the condition that $\bar{\partial}\omega = 0$. So in fact we have an exact sequence

$$0 \longrightarrow \Omega^p(X) \longrightarrow \mathcal{A}^{p,0}(X) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}(X) \longrightarrow \dots$$

We will see that in fact, $H^{a,b}(X) = H^b(\Omega_X^a)$, where Ω_X^a is the sheaf (space) of holomorphic a forms and H^b is the cohomology of the sheaf.

1.3 Sheaf Cohomology

1.3.1 Sheaves

We recall the definition of a sheaf.

Definition 2. If X is a complex manifold, a sheaf \mathcal{F} on X consists of, for each open $U \subseteq X$, an abelian group (or ring, or any desired object) $\mathcal{F}(U)$ (also denoted $\Gamma(\mathcal{F}, U)$ and called the set of sections over U), and a collection of restriction maps such that for each $U \subseteq V \subseteq X$, $r_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$, and which satisfy:

1. Identity: $r_{U,U} = 1_{\mathcal{F}(U)}$.
2. Compatibility: If $U \subseteq V \subseteq W \subseteq X$, then

$$\begin{array}{ccc} \mathcal{F}(W) & \xrightarrow{r_{W,V}} & \mathcal{F}(V) \\ & \searrow r_{W,U} & \downarrow r_{V,U} \\ & & \mathcal{F}(U) \end{array}$$

commutes. We will often denote $r_{V,U}(\sigma)$ by $\sigma|_U$, for $\sigma \in \mathcal{F}(V)$.

3. Sheaf axiom (gluing): Let $U = \bigcup_{\alpha} U_{\alpha}$. If $\sigma_{\alpha} \in \mathcal{F}(U_{\alpha})$ and $\sigma_{\alpha}|_{U_{\alpha} \cap U_{\beta}} = \sigma_{\beta}|_{U_{\alpha} \cap U_{\beta}}$ for all α, β , then there exists a (unique) $\sigma \in \mathcal{F}(U)$ such that $\sigma_{\alpha} = \sigma|_{U_{\alpha}}$ for all α .

Let's consider a few familiar examples, each with the natural sections and restriction maps

Example 2. \mathcal{C}_X^{∞} , the sheaf of smooth functions on X .

Example 3. \mathcal{O}_X , the sheaf of holomorphic functions on X .

Example 4. Ω_X^p , the sheaf of holomorphic p -forms on X .

Example 5. \mathcal{A}_X^n , the sheaf of n -forms on X .

Example 6. $\mathcal{A}_X^{a,b}$, the sheaf of (a,b) -forms on X .

A different kind of example:

Example 7. The skyscraper sheaf \mathbb{C}_p given by

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

along with the natural restriction maps.

1.3.2 Sheaf morphisms

Now that we have the notion of a sheaf, let's discuss maps between sheaves.

Definition 3. Let \mathcal{F}, \mathcal{G} be sheaves on X . Then a morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$ consists of morphisms $\phi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for all open $U \subseteq X$ such that the ϕ_U commutes with the restriction maps for \mathcal{F} and \mathcal{G} , that is, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \end{array}$$

for all $U \subseteq V$ open in X , where the maps on the left and right are the appropriate restriction maps.

Example 8. Let \mathcal{O}_X^* be the sheaf of nowhere vanishing holomorphic functions on X . It is a subsheaf of \mathcal{O}_X . Then

$$\begin{aligned} \exp : \mathcal{O}_X &\rightarrow \mathcal{O}_X^* \\ f &\mapsto e^{2\pi i f} \end{aligned}$$

is a sheaf morphism.

1.3.3 Kernel and Image

Given a morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$, we wish to define $\ker \phi$ and $\text{im } \phi$. For $\ker \phi$, the naïve definition works and defines the subsheaf $\ker \phi \subseteq \mathcal{F}$:

$$(\ker \phi)(U) := \ker\{\phi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)\}.$$

This construction in fact satisfies the sheaf axioms. If one were to try the naïve definition for the image of ϕ , however, namely,

$$(\text{im } \phi)(U) \stackrel{?}{=} \text{im}\{\phi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)\}$$

this construction would *not*, in general, be a sheaf. It is a *presheaf*, that is, it satisfies axioms 1 and 2, but not the 3rd sheaf axiom (gluing). Let's consider an example to see how this works. Suppose X is a compact complex manifold, and let $p, q \in X$ be distinct points. Note that the direct sum of skyscraper sheaves, $\mathbb{C}_p \oplus \mathbb{C}_q$, formed in the natural way by taking the direct sum of each section, is also a sheaf. Then we have a natural morphism

$$\phi : \mathcal{O}_X \rightarrow \mathbb{C}_p \oplus \mathbb{C}_q$$

given by evaluating each function at p and q , specifically,

$$\begin{aligned} \phi_U : \mathcal{O}_X(U) &\rightarrow \mathbb{C}_p \oplus \mathbb{C}_q \\ f &\mapsto \begin{cases} (f(p), f(q)) & \text{if } p, q \in U \\ (f(p), 0) & \text{if } p \in U, q \notin U \\ (0, f(q)) & \text{if } p \notin U, q \in U \\ (0, 0) & \text{if } p, q \notin U \end{cases} \end{aligned}$$

Now, take $U = X \setminus \{q\}$ and $V = X \setminus \{p\}$. Then $\phi_U : \mathcal{O}_X(U) \rightarrow \mathbb{C} \oplus 0$ by $f \mapsto (f(p), 0)$ and if $a \in \mathbb{C}$, then $(a, 0) \in \text{im } \phi_U$ (consider the constant function $f(z) = a$). Similarly, $\phi_V : \mathcal{O}_X(V) \rightarrow 0 \oplus \mathbb{C}_q$ by $f \mapsto (0, f(q))$, so for any $b \in \mathbb{C}$, $(0, b) \in \text{im } \phi_V$. Note that $\mathcal{O}_X(V)$ may include functions with poles at p , just as $\mathcal{O}_X(U)$ may for q . Now, if we take $a \neq b$, then there does not exist an $f \in \mathcal{O}_X(X)$ such that

$$\phi_X(f) = \begin{cases} (a, 0) & \text{at } p \\ (0, b) & \text{at } q \end{cases}$$

because $\mathcal{O}_X(X)$ only consists of constant functions, since X is compact.

The solution, as you might expect if you are familiar with sheaves, is to *sheafify* the presheaf we obtain through our naïve definition of the image of ϕ . This results in the correct definition:

Definition 4. Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on X . We define the image sheaf of ϕ by

$$(\text{im } \phi)(U) = \{s \in \mathcal{G}(U) : \forall p \in U, \exists V \subseteq U \text{ s.t. } p \in V \text{ and } s|_V \in \text{im}(\phi_V)\}.$$