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1.1 Weierstrass $\wp$-function

Suppose $\Lambda \subset \mathbb{C}$ is a lattice. Last time we defined the Weierstrass $\wp$-function

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right).$$

Then we defined a map $\phi(z) : \mathbb{C}/\Lambda \to \mathbb{P}^2$, $z \mapsto [z^3 : z^3\wp(z) : z^3\wp'(z)]$. This map is well-defined and $\phi(0) = [0 : 0 : 1]$ and both $\wp$ and $\wp'$ are holomorphic at any other point.

Lemma 1.1. *$\phi : \mathbb{C}/\Lambda \to \mathbb{P}^2$ is an embedding.*

Proof. We have to show that ϕ is injective and $\phi'(z) \neq 0$. Suppose $\phi(z_1) = \phi(z_2)$ for some points z_1 and z_2, i.e. $\wp(z_1) = \wp(z_2)$ and $\wp'(z_1) = \wp'(z_2)$. Since $\wp(z)$ has only two poles (counted with multiplicities) in any fundamental parallelogram, hence the same is true for the function $\eta(z) := \wp(z) - \wp(z_1)$. Consider two cases:

1) $2z_1 \notin \Lambda = \lambda_1\mathbb{Z} + \lambda_2\mathbb{Z}$. Hence $z_1 \neq -z_1 \ mod\ \Lambda$. Since $\wp(z)$ is an even function we obtain $\eta(z_2) = \eta(z_1) = \eta(-z_1) = 0$. But $\eta(z)$ has exactly 2 zeroes with multiplicity (as $\wp$ does). This shows that if $z_1 \neq z_2 \ mod\ \Lambda$, then $z_2 = -z_1 \ mod\ \Lambda$. Now $\wp'(z)$ is odd, so we have $\wp'(z_2) = \wp'(z_1) = -\wp'(-z_1) = -\wp'(z_2)$. Thus $\wp'(z_1) = \wp'(z_2) = 0$. But all the zeroes of $\wp'(z)$ are $\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_1+\lambda_2}{2}$ (it is easy to check that these are zeroes and because $\wp'$ has a single pole of order 3 in any fundamental parallelogram, it has exactly three zeroes). Now the values of $\wp$ at these three points are all distinct, which is a contradiction, showing that $z_1 = z_2$.

2) $2z_1 \in \Lambda$. Again, since $\wp'(z)$ is odd $\eta'(z_1) = \wp'(z_1) = 0$, so $\eta(z_1) = \eta'(z_1) = \eta(z_2) = 0$, but η has only two zeroes and hence $z_1 = z_2 \ mod\ \Lambda$.

Show that $\phi'(z) \neq 0$ for exercise. ☐

Unfortunately this idea for embedding tori in projective spaces using periodic functions with respect to the lattice, doesn't generalize to higher dimensions because such functions will necessarily be meromorphic and unlike in dimension 1 we can no longer eliminate the poles.

1.2 Theta functions

For embedding a torus in a projective space we don't actually need periodic functions. It's enough to use so called quasi-periodic functions. For simplicity let the dimension be 1 and the lattice be $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H}_1 = \{z \in \mathbb{C} : Im z > 0\}$.

Definition 1. *A quasi-periodic function with respect to the lattice Λ_τ is a holomorphic function f, such that*

$$f(z+1) = f(z), \forall z \in \mathbb{C}$$
$$f(z+\tau) = e^{az+b} f(z), \forall z \in \mathbb{C}$$

for some $a, b \in \mathbb{C}$.

It's easy to see that having quasi-periodic functions $f_0, f_1, \dots, f_n$ with the same a, b the map $C/\Lambda_\tau \to \mathbb{P}^n$, $z \mapsto [f_0(z), f_1(z), \dots, f_n(z)]$ is well defined ($\phi(z+\lambda) = \phi(z)$) and thus it is a map from the torus to the projective space.

Now the question is are there any quasi-periodic functions (also called theta functions). Riemann was the first to construct one. Here is the well known Riemann theta function.

$$\vartheta : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$$

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}$$

The following computation shows that $\vartheta(\cdot, \tau)$ is quasi-periodic with respect to Λ_τ:

$$\vartheta(z+\tau, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z + 2\pi i n \tau} = e^{-\pi i \tau - 2\pi i z} \sum_{n \in \mathbb{Z}} e^{\pi i (n+1)^2 \tau + 2\pi i (n+1) z} = e^{-\pi i \tau - 2\pi i z} \vartheta(z, \tau)$$

$$\vartheta(z+1, \tau) = \vartheta(z, \tau)$$

Thus $\vartheta(z, \tau)$ gives a family of quasi-periodic functions, parameterized by τ, in other words the embedding that we'll get is not only an embedding of a single curve, but of all of them, so we will get an embedding of the moduli space in some $\mathbb{P}^n$.

Now we have at least one example. But we need more theta functions. We can use a simple trick to construct more out of the one that we have.

Definition 2. *Theta function of characteristic* $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, $a, b \in \mathbb{R}$ *is the following function:*

$$\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z, \tau) := \sum_{n \in \mathbb{Z}} e^{\pi i (a+n)^2 \tau + 2\pi i (n+a)(z+b)}$$

Exercise 1.2. *Prove that* $\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z, \tau)$ *is quasi-periodic.*

1.3 Embedding of tori using theta functions

Fix $l \geq 2, l \in \mathbb{Z}$ and consider the group $\frac{1}{l}\mathbb{Z} \times \frac{1}{l}\mathbb{Z}/\mathbb{Z} \times \mathbb{Z}$ of torsion points of order l on an elliptic curve. Also fix a set of l^2 representatives $\{(a_i, b_i)\}$ for this group and define the map:

$$\phi_l : \mathbb{C}/\Lambda_\tau \to \mathbb{P}^{l^2-1}$$

$$z \mapsto [\vartheta\left[\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right](lz, \tau) : \dots : \vartheta\left[\begin{smallmatrix} a_{l^2-1} \\ b_{l^2-1} \end{smallmatrix}\right](lz, \tau)]$$

Check that $\phi_l(z)$ is a well-defined map. It is also an embedding for $l \geq 2$. Later we will view these functions as sections of line bundles.

Fact 1.3. *When* $l = 2$ *we have an embedding of the curve in* $\mathbb{P}^3$*, which is of degree four. This is the smallest possible degree due to the theorem of Riemann-Roch.*

1.4 Cohomology of complex tori

Let again $V = \mathbb{C}^g$ and $\Lambda \simeq \mathbb{Z}^{2g} \subset V$ be a full lattice, $X = V/\Lambda$. We want to compute the de Rham cohomology H^n_{dR}. We know that there is a canonical isomorphism for the first singular homology group: $H_1(X,\mathbb{Z}) \simeq \Lambda$, $\lambda \mapsto [\overline{\lambda}]$, where $\overline{\lambda} : [0,1] \to X$, $\overline{\lambda}(t) = t\lambda$. In particular the singular homology group is free, which implies

$$H^1(X,\mathbb{Z}) = H_1(X,\mathbb{Z})^\vee = \Lambda^\vee = Hom_{\mathbb{Z}}(\Lambda,\mathbb{Z})$$

Topologically $X \simeq (S^1)^{2g}$ and we can use the Kunneth theorem to get:

$$H^n(X,\mathbb{Z}) \simeq \wedge^n H^1(X,\mathbb{Z}) \simeq \wedge^n \Lambda^\vee \simeq Alt^n_{\mathbb{Z}}(\Lambda,\mathbb{Z})$$

which is the alternating form in n variables and coefficients in $\mathbb{Z}$.

Lemma 1.4. $H^n_{dR}(X,\mathbb{C}) \simeq Alt^n_{\mathbb{R}}(V,\mathbb{C})$.

Proof. Since $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq V$ we have $Alt^n_{\mathbb{Z}}(\Lambda,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \simeq Alt^n_{\mathbb{R}}(V,\mathbb{R})$. Now

$$Alt^n_{\mathbb{Z}}(\Lambda,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq Alt^n_{\mathbb{Z}}(\Lambda,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \simeq Alt^n_{\mathbb{R}}(V,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq Alt^n_{\mathbb{R}}(V,\mathbb{C})$$

Note that here we identify V with the $V \otimes_{\mathbb{R}} \mathbb{C}$ - the complexified V. The de Rham theorem gives an isomorphism between singular and de Rham cohomology, namely

$$H^n_{dR}(X,\mathbb{C}) \simeq H^n(X,\mathbb{C}) \simeq H^n(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq Alt^n_{\mathbb{R}}(V,\mathbb{C})$$

□

Recall some of the properties of the de Rham cohomology. If X is $\mathbb{R}$-manifold of dimension $2g$, $\mathbf{A}^n(X)$ the space of n-forms on X, then in local coordinates $x_1, x_2, \ldots, x_{2g}$, $\omega \in \mathbf{A}^n(x)$ is written as $\omega = \sum a_{i_1,\ldots,\, i_n} dx_{i_1} \wedge \cdots \wedge dx_{i_n}$, $a_{i_1,\ldots,\, i_n} \in C^\infty(X,\mathbb{C})$. The operator $d_n : \mathbf{A}^n(X) \to \mathbf{A}^{n+1}(X)$ acts as

$$d_n\omega = \sum_{i_1<\cdots<i_n} \sum_{j=1}^{2g} \frac{\partial a_{i_1,\ldots,\, i_n}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}$$

It is easy to check that $d_n \circ d_{n-1} = 0$ and thus $H^n_{dR}(X,\mathbb{C}) := Ker\ d_n / Im\ d_{n-1}$.

What happens on our tori? Fix a basis $\lambda_1, \ldots, \lambda_{2g}$ of $\Lambda = H_1(X,\mathbb{Z})$ and denote by $x_1, \ldots, x_{2g}$ the corresponding real coordinate functions on V. Obviously the differentials $dx_1, \ldots, dx_n$ are invariant (under translations) 1-forms on V. Hence every dx_i is the pullback of a uniquely determines invariant 1-form on X via $\pi : V \to X$. We denote these forms on X also by dx_i. Now cohomology classes $[dx_i]$ are basis of $H^1(X,\mathbb{C})$, because by construction $\int_{\lambda_i} dx_j = \delta_{ij}$. We have

$$Alt^n_{\mathbb{R}}(V,\mathbb{C}) = \bigwedge^n Hom_{\mathbb{R}}(V,\mathbb{C}) \simeq \bigwedge^n H^1(X,\mathbb{C})$$

Thus the homology classes of n-forms $dx_{i_1} \wedge \cdots \wedge dx_{i_n}$, $i_1 < \cdots < i_n$ form a basis of $H^n(X,\mathbb{C})$. We proved the following

Lemma 1.5. $\forall \omega \in H^n_{dR}(X, \mathbb{C})$ *is uniquely represented by a translation invariant n-form with complex coefficients.*

At the end we will give a prove of the following important theorem.

Theorem 1.6. *The holomorphic tangent bundle of a torus is trivial.*

Proof. $X = V/\Lambda$ and $\pi : TX \to X$ is the projection map. To show that the bundle is trivial it is enough to provide the isomorphism ϕ, such that the diagram commutes

$$\begin{tikzcd}
X \times V \arrow[r, "\phi", "\sim"'] \arrow[dr, "pr"'] & TX \arrow[d, "\pi"] \\
& X
\end{tikzcd}$$

We have a canonical identification $T_0(X) \simeq V$. Let $t_x : X \to X$ be the translation map $a \mapsto a + x$. Then define ϕ as the map $(x, \xi) \mapsto (dt_x)_0(\xi)$. This is well-defined bundle map and is isomorphism, because $(dt_x)_0 : T_0(X) \to T_x(X)$ is. ☐

Corollary 1.7. *The vector bundle Ω^n_X of holomorphic n-forms on X is trivial for every $n \geq 1$.*