1 Abelian Varieties, Feb 7 2006 - Notes taken by Dave Jensen

1.1 Natural Line Bundles on Abelian Varieties

If X is any projective variety and $i : X \to \mathbb{P}^n$ is the inclusion map, then letting $L = i^*\mathcal{O}_{\mathbb{P}^n}(1)$, we get a line bundle $L \to X$. Moreover, if we are given non-vanishing sections $s_0, s_1, \dots, s_n \in \Gamma(X, L)$, then we can recover the inclusion i by defining $i(p) = [s_0(p), s_1(p), \dots s_n(p)]$ for all $p \in X$. Conversely, given any line bundle on a variety $L \to X$ and sections $s_0, s_1, \dots, s_n(p) \in \Gamma(X, L)$ which do not vanish simultaneously at any point, we can always define a map from X to $\mathbb{P}^n$ by setting

$$f(p) = [s_0(p), \dots s_n(p)]$$

for each point $p \in X$.

Recall that an abelian variety X is a quotient of a vector space V by a lattice Λ, that is, $\pi : V \to X = V/\Lambda$. Since all line bundles over V are trivial, if we have a line bundle $L \to X$, then π^*L is isomorphic to the trivial line bundle $V \times \mathbb{C}$. For every $z \in V$, therefore, we get an isomorphism of vector spaces $\chi_z : L(\pi(z)) \to \pi^*L(\pi(z))$. Since for every $\lambda \in \Lambda$, $\pi(z) = \pi(z + \lambda)$, we have another isomorphism $\chi_{z+\lambda} : L(\pi(z)) \to \pi^*L(\pi(z + \lambda))$. There must therefore be an isomorphism from $\pi^*L(\pi(z))$ to $\pi^*L(\pi(z+\lambda))$, given by multiplication by a complex number $e_\lambda(z)$. From this, we can see that $\chi_{z+\lambda} = e_\lambda(z)\chi_z$ and $e_{\lambda+\lambda'}(z) = e_\lambda(z)e_{\lambda'}(z + \lambda)$. In addition, if we are given the e_λ's, we can recover L.

1.2 Group Cohomology

Throughout this discussion, G is a group and M is a G-module. We define the p-th chain group $C^p(G, M) = \{f : \Pi_{i=1}^p G \to M\}$. The chain complex is then the sequence

$$0 \to C^0(G, M) \overset{d^0}{\to} C^1(G, M) \overset{d^1}{\to} \cdots \overset{d^{p-1}}{\to} C^p(G, M) \overset{d^p}{\to} \cdots$$

with the maps d^p defined as follows:

$$(d^p f)(g_0, g_1, \dots, g_p) := g_0 f(g_1, \dots g_p) - f(g_0, g_1 g_2, g_3, \dots, g_p) + \dots$$

$$+(-1)^p f(g_0, \dots g_{p-2}, g_{p-1}g_p) + (-1)^{p+1} f(g_0, g_1, \dots, g_{p-1}$$

It is straightforward to check that $d^p \circ d^{p-1} = 0$. We then define the cohomology group of G with coefficients in M to be $H^p(G, M) := H^p(C^{\cdot}(G, M))$. As an example, we note that $C^0(G, M) = M$, and $d^0(m)(g) = gm - m$. It follows that $H^0(G, M)$ is precisely the set of elements of M that are fixed by the action of G.

If we have a map of modules $M \to N$, then we have an induced map between in cohomology $H^p(G, M) \to H^p(G, N)$. In addition, if we have an exact sequence

$$0 \to M \to N \to P \to 0,$$

then we have an induced exact sequence

$$0 \to C^{\cdot}(G, M) \to C^{\cdot}(G, N) \to C^{\cdot}(G, P) \to 0.$$

We then get an induced sequence in cohomology

$$0 \to H^0(G,M) \to H^0(G,N) \to H^0(G,P) \to H^1(G,M) \to \cdots \to H^p(G,P) \to H^{p+1}(G,M) \to \cdots$$

1.3 The First Chern Class

Now, we want to use group cohomology to understand the Picard group of an abelian variety X. To do this, we first define a Λ-module structure on $\mathbb{Z}$, $\mathcal{O}_X$, and $\mathcal{O}_X^*$. The action of Λ on $\mathbb{Z}$ is trivial, and the action on the other two groups is given by $\lambda f(z) = f(z+\lambda)$.

Notice that, for an arbitrary complex manifold X, we have a short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \overset{e^{2\pi i}}{\to} \mathcal{O}_X^* \to 0.$$

Then the induced sequence in Cech cohomology is

$$H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \overset{c_1}{\to} H^2(X,\mathbb{Z}).$$

If $L \to X$ is a line bundle, then $c_1(L) \in H^2(X,\mathbb{Z})$, with c_1 as defined above, is the first Chern class of L. The first Chern class is the only C^∞ (topological) invariant of a line bundle. This follows because we have previously seen that if $\mathbb{C}_X^\infty$ denotes the sheaf of C^∞ functions on X, we can use the existence of partitions of unity to show that $H^1(X, C_X^\infty) = H^2(X, C_X^\infty) = 0$.

For a complex torus $X = V/\Lambda$, we have that $\mathrm{Pic}(X) = H^1(X,\mathcal{O}_X^*) = H^1(\Lambda, H^0(\mathcal{O}_V^*))$. We also have that $H^2(X,\mathbb{Z}) = Alt^2_{\mathbb{Z}}(\Lambda,\mathbb{Z})$, so the first Chern class can be thought of as a map

$$c_1 : H^1(\Lambda, H^0(\mathcal{O}_V^*)) \to Alt^2_{\mathbb{Z}}(\Lambda,\mathbb{Z}).$$

We would like to show that $H^2(\Lambda,\mathbb{Z}) = Alt^2_{\mathbb{Z}}(\Lambda,\mathbb{Z})$.

Lemma 1.1. *There is a canonical isomorphism of groups*

$$\alpha : H^2(\Lambda,\mathbb{Z}) \to Alt^2_{\mathbb{Z}}(\Lambda,\mathbb{Z})$$

given by $\alpha : F(\lambda,\mu) \mapsto F(\lambda,\mu) - F(\mu,\lambda)$.

Proof. We need to show that this map is bijective. Notice that a 2-cocycle in $C^2(\Lambda,\mathbb{Z})$ is a map $F : \Lambda \times \Lambda \to \mathbb{Z}$ such that, for all $\lambda_0, \lambda_1, \lambda_2 \in \Lambda$, the following holds:

$$F(\lambda_0,\lambda_2) - F(\lambda_0+\lambda_1,\lambda_2) + F(\lambda_0,\lambda_1+\lambda_2) - F(\lambda_0,\lambda_1) = 0. \qquad (*)$$

Now, let $f, g \in \Lambda^\vee$. Then $f \otimes g$ is a 2-cocycle, so it satisfies (*). Thus, $\alpha(f\otimes g) = f(\lambda)g(\mu) - f(\mu)g(\lambda) = f \wedge g$. Of course, elements of the form $f \wedge g$ generate $Alt^2_{\mathbb{Z}}(\Lambda,\mathbb{Z})$, so F is surjective.

We did not show injectivity in class, but I'm going to put a little box here anyway.

□

We now turn to the following proposition. If $\pi : V \to X = V/\Lambda$ is a complex torus and F is any sheaf on X, then there exist functorial group homomorphisms $\phi : H^p(\Lambda, H^0(\pi^* F)) \to H^p(X,F)$ such that, if $H^i(\pi^* F) = 0$ for $i \geq 1$, then the maps ϕ are isomorphisms.

Proof. Note that we have already done the case where $p = 1$ and $F = \mathcal{O}_X^*$. Let $\{U_i\}$ be an open cover of X. Then $\pi^{-1}(U_i) = \coprod_{\lambda\in\Lambda}(V_i + \lambda)$ for some open set $V_i \subseteq V$. Thus, if $U_{ij} = U_i \cap U_j \neq \varnothing$, addition by λ_{ij} gives an isomorphism from $\pi_i^{-1}(U_{ij})$ to itself. This helps us establish isomorphisms between π^*F and F.

Now, let $f \in H^p(\Lambda, H^0(\pi^*F))$. We then define $\phi(f)_{ij} = (\pi_{ij}^*)^{-1}(f(\lambda_{ij}))|_{\pi_i^{-1}(U_{ij})} \in F(U_{ij})$. In order for this to make sense, we must check that $\phi(f)_{ij} + \phi(f)_{jk} = \phi(f)_{ik}$ on U_{ijk}, but this not a difficult exercise. □

Now, for a torus $X = V/\Lambda$, we have the isomorphisms

$$H^1(\Lambda, H^0(\mathcal{O}_V^*)) \xrightarrow{\delta} H^2(\Lambda, \mathbb{Z}) = \mathrm{Alt}^2_{\mathbb{Z}}(\Lambda, \mathbb{Z}),$$

and

$$H^2(\Lambda, \mathbb{Z}) \xrightarrow{\alpha} H^2(X, \mathbb{Z})$$

established in the earlier proposition. By collecting all these facts, we can explicitly compute the first Chern class of a line bundle on a torus.

Theorem 1.2. *The first Chern class map on a torus $c_1 : H^1(\Lambda, H^1(\mathcal{O}_V^*)) \to Alt^2_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ is given by the following formula*

$$c_1(\{e_\lambda\}) := f_\lambda(z) - f_\mu(z) - f_\lambda(z + \mu) + f_\mu(z + \lambda),$$

where $e_\lambda = e^{2\pi i f_\lambda}$, and $z \in V$ is an arbitrary point.

Proof. Since the map $e^{2\pi i} : C^1(\Lambda, H^0(\mathcal{O}_V)) \to C^1(\Lambda, H^0(\mathcal{O}_V^*))$ is surjective, we may choose liftings f_λ such that $e^{2\pi i f_\lambda} = e_\lambda$. Then $d\{f_\lambda\}(\lambda, \mu) = \lambda f(\mu) - f(\lambda\mu) + f(\lambda) = \lambda f_\mu - f_{\lambda+\mu} + f_\lambda$. After symmetrizing, we obtain that

$$c_1(\{e_\lambda\}) = \lambda f_\mu(z) - f_{\lambda+\mu}(z) + f_\lambda(z) - \mu f_\lambda(z) + f_{\lambda+\mu}(z) - f_\mu(z) = f_\lambda(z) - f_\mu(z) - f_\lambda(z+\mu) + f_\mu(z+\lambda).$$

□