ABELIAN VARIETIES: 16 FEBRUARY 2006, NOTES TAKEN BY ALEX KAHLE

1. $\mathrm{Pic}^0(X)$ as a torus

Recall $\mathrm{Pic}^0(X)$ is the space of all topologically trivial line bundles, that is, line bundles with trivial first chern class. All such line bundles are of the form $L(0, \chi) \to X$, with χ a character. The total space of such a line bundle is

$$L(0, \chi) = V \times \mathbb{C} / \{(z, t) \sim (z + \lambda, \chi(\lambda) t)\}.$$

Lemma 1.1. *$\bar{\Omega} \tilde{\to} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, with the isomorphism being given by $l \mapsto \Im(l)$.*

Proof. One explicitly computes the inverse: for $\phi : V \to \mathbb{R}$, define

$$l(z) = -\phi(iz) + i\phi(z).$$

One may easily check that $l(z)$ defined this way is indeed an element of $\bar{\Omega}$. □

Proposition 1.2. *There is an isomorphism of tori $\mathrm{Pic}^0(X) \simeq \bar{\Omega}/\hat{\Lambda}$, where $\hat{\Lambda} = \left\{ l \in \bar{\Omega} : \ \mathrm{Im}\, l(\Lambda) \subseteq \mathbb{Z} \right\}$.*

Proof. Recall $\mathrm{Pic}^0(X) \simeq \mathrm{Hom}\left(\Lambda, S^1\right)$ canonically by the Appel-Humbert theorem. We define a map $\rho : \bar{\Omega} \to \mathrm{Hom}\left(\Lambda, S^1\right)$ by

$$l \mapsto e^{2\pi i \Im(l(\lambda))}.$$

It is immediate that ρ is surjective, and

$$\ker(\rho) = \left\{ l \in \bar{\Omega} : \ \mathrm{Im}\, l(\Lambda) \subseteq \mathbb{Z} \right\} = \hat{\Lambda}.$$

The isomorphism follows. □

Definition 1.3 (The dual torus)**.** The *dual torus* of a torus X is denoted by $\hat{X}$, and is defined to be $\hat{X} := \mathrm{Pic}^0(X)$, which is realised as a torus by the previous proposition.

Remark 1.4. The identification of $\hat{X}$ with $\bar{\Omega}/\hat{\Lambda}$ gives it a natural complex structure.

Remark 1.5. Taking the dual of a torus is functorial. Given $\varphi : X \to Y$, a homomorphism of tori, there exists a $\hat{\varphi} : \hat{Y} \to \hat{X}$, defined as follows. Write $X = V/\Lambda$, $Y = V'/\Lambda'$, so that φ has an analytic representation $f : V \to V'$. Then one may simply take $\hat{\varphi}$ to be the map induced by f^* (the map $f^* : l \mapsto l \circ f$).

We will now explore the relation between a torus and its dual. If $X = V/\Lambda$, and $L \in \mathrm{Pic}(X)$, one may define an associated map $\Phi_L : X \to \hat{X}$ via[1]

$$\Phi_L(x) = t_x^*(L) \otimes L^\vee \in \hat{X}.$$

[1] If $L \to X$ is a line bundle over a variety, then $L^\vee = \mathrm{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X)$. More concretely, if L is given by the transition functions $\{g_{ab}\} \in H^1(X, \mathcal{O}_X^*)$, then $L^\vee$ has transition functions $\{g_{ab}^{-1}\}$. It is often written L^{-1}, and is indeed the inverse of L in $\mathrm{Pic}(X)$.

If X is a torus, and $L = L(H, \chi)$, then $L^\vee = L\left(-H, \chi^{-1}\right)$.

Proposition 1.6. *$\Phi_L : X \to \hat{X}$ is a homomorphism of tori.*

Proof. We are required to show $\Phi_L(x+y) = \Phi_L(x) \otimes \Phi_L(y)$. But this is simply the *theorem of the square.* □

We shall now find the analytic representation of Φ_L. Let us choose a representative $z \in V$ for $x \in X$, i.e. $\pi(z) = x$, and write $L = L(H, \chi)$. Then

$$\begin{aligned} \Phi_L(x) &= t_x^*(L(H, \chi)) \otimes L(H, \chi)^\vee \\ &= L\left(H, \chi e^{2\pi i E(z, \cdot)}\right) \otimes L\left(-H, \chi^{-1}\right) \\ &= L\left(0, e^{2\pi i E(z, \cdot)}\right) \\ &= L\left(0, e^{2\pi i \Im(H(z, \cdot))}\right). \end{aligned}$$

Thus,

Proposition 1.7. *The analytic representationof Φ_L is given by:*

$$\begin{aligned} \rho_a(\Phi_L) : V &\to \bar{\Omega} \\ z &\mapsto H(z, \cdot) \in \bar{\Omega}. \end{aligned}$$

Corollary 1.8. *Φ_L depends only on H, so line bundles in the same topological class give the same map.*

Corollary 1.9. *Notice $\Phi_{L \otimes L'} = \Phi_L + \Phi_{L'}$.*

Corollary 1.10. *Φ_L is an isogeny when H is non-degenerate (or equivalently, E is non-degenerate). By extension, a line bundle is called* non-degenerate *when its first chern class is non-degenerate.*

Definition 1.11. Given $L \in \mathrm{Pic}(X)$, define the group

$$k(L) = \{x \in X : t_x^* L = L\}.$$

Notice $K(L) =: \ker(\Phi_L) = \Lambda(L)/\Lambda$, where

$$\Lambda(L) = \rho_a(\Phi_L)^{-1}(\hat{\Lambda}) = \{z \in V : \mathrm{Im}\, H(z, \Lambda) \subseteq \mathbb{Z}\}.$$

Corollary 1.12. *L is non-degenerate iff $|K(L)| < \infty$.*

Exercise 1.13. *Show that* $\deg(\Phi_L) = \det(Im(H))$.

2. The Poinceré bundle

2.1. **Praeludium: a motivation.** One may define a contravariant functor associated to a complex variety X, called the *Picard functor*, from the category of analytic spaces (which one may think of simply as complex varieties), to the category of sets:

$$\mathrm{Pic}_X : \{\text{analytic spaces}\} \to \{\text{sets}\}.$$

In order to define this, one must introduce an equivalence relation on line bundles on $X \times T$, for any analytic space T. We say $L \sim L'$ (where these are line bundles on $X \times T$) when $L = L' \otimes \pi^* M$, where $M \to X$ is a line bundle on X, and $\pi : X \times T \to X$ is projection on the first factor. Then, on objects,

$$\mathrm{Pic}_X(T) = \{\text{line bundles on } X \times T\}/\sim,$$

and on morphisms

$$\mathrm{Pic}_X(\phi : T \to S) = L \to (\mathrm{Id} \otimes \phi)^*(L).$$

A natural question to ask when a functor arises geometrically is when it is representable. In other words, we wish to know (in the case of Pic_X), when we can find a variety (suggestively called $\mathrm{Pic}(X)$) such that there is a natural transformation from $\mathrm{Pic}_X \to \mathrm{Hom}\,(\cdot,\, \mathrm{Pic}(X))$. In other words, given a $\phi : T \to S$, we wish to have

$$\begin{array}{ccc} \mathrm{Pic}_X(S) & \longrightarrow & \mathrm{Hom}\,(S,\, \mathrm{Pic}(X)) \\ \Big\downarrow{\scriptstyle (\mathrm{Id}\otimes\phi)^*} & & \Big\downarrow{\scriptstyle \phi^*} \\ \mathrm{Pic}_X(T) & \longrightarrow & \mathrm{Hom}\,(T,\, \mathrm{Pic}(X)) \end{array}.$$

In particular, supposing Pic_X be representable, let us take $S = \mathrm{Pic}(X)$ in the above diagram. One then obtains an isomorphism from $\mathrm{Hom}\,(\mathrm{Pic}(X),\, \mathrm{Pic}(X))$ to line bundles on $\mathrm{Pic}(X)$. The image of the identity morphism by this isomorphism is called the *Poincaré bundle*, and is denoted $\mathscr{L}$. This has the property that, given an $L \in \mathrm{Pic}(X)$, and defining $j_L : X \hookrightarrow X \times \mathrm{Pic}(X)$ by $x \to (x,\, L)$, one has $j_L^* \mathscr{L} = L$.

2.2. **The Poincare bundle on tori.**

We now take X to be a torus. We wish to construct the Poincaré bundle on $X \times \hat{X}$. This will be a line bundle $\mathscr{L} \to X \times \hat{X}$ satisfying

(1) For every $L \in \hat{X}$, $j_L^*(\mathscr{L}) = L \in \mathrm{Pic}^0(X)$, where $j_L : X \hookrightarrow X \times \hat{X}$ is given by $j_L(x) = (x,\, L)$.
(2) $\mathscr{L}|_{\{0\}\times\hat{X}}$ is trivial.

The second condition is not essential, but ensures uniqueness in the choice of $\mathscr{L}$.

As $X \times \hat{X}$ is simply a torus, we may write $\mathscr{L} = L\,(H,\, \chi)$, and reduce the problem to determining H and χ. Let us recall the structure of $X \times \hat{X}$ as a torus: $X \times \hat{X} = (V \times \bar{\Omega})/(\Lambda \times \hat{\Lambda})$. Thus $H : (V \times \bar{\Omega}) \times (V \times \bar{\Omega}) \to \mathbb{C}$. One possible choice for such an H is

$$H((z_1,\, l_1),\, (z_2,\, l_2)) = l_1(z_2) + \overline{l_2(z_1)}.$$

This is certainly hermitian, and $\mathrm{Im}\, H((\Lambda \times \hat{\Lambda}) \times (\Lambda \times \hat{\Lambda})) \subseteq \mathbb{Z}$, by the definition of $\hat{\Lambda}$.

The semicharacter χ must be a map $\chi : \Lambda \times \hat{\Lambda} \to S^1$. A candidate for this is $\chi(\lambda,\, l) = e^{\pi i \Im(l(\lambda))}$. A line bundle $L\,(H,\, \chi)$ with this choice of first chern class and semi-character has factors of automorphy

$$a_{(\lambda,\, l_0)}(z,\, l) = e^{\pi i \mathrm{Im}(l_0(\lambda))} e^{\pi l(\lambda) + \pi \overline{l_0(z)} + \pi \mathrm{Re}(l_0(\lambda))}.$$

Theorem 2.1. *The Poincaré bundle of a torus X is given by $\mathscr{L} = L(H, \chi)$, where H and χ are defined as above. The bundle defined this way satisfies the properties (1) and (2) defined at the beginning of the discussion.*

Proof. Choose $L \in \hat{X}$. It may be written $L = L(0,\, e^{2\pi i \mathrm{Im}(l_1)})$ for a given l_1 in $\bar{\Omega}$ (recall that line bundles in $\hat{X}$ have $H = 0$). As j_L (defined above) is a map of tori, it factors as a composition of a translation and a homomorphism, that is, $j_L = t_{(0,\, L)} \circ j_0$. Acting on the universal covers, the translation is translation by

$(0,\, l_1)$. Thus

$$\begin{aligned} j_L^*(\mathscr{L}) &= j_0^*\left(t_{(0,\, l_1)}^*(\mathscr{L})\right) \\ &= j_0^*\left(L\left(H,\, \chi e^{2\pi i \mathrm{Im}\, H((0,\, l_1),\, \cdot)}\right)\right) \\ &= L\left(0,\, e^{2\pi i \mathrm{Im}(l_1(\lambda))}\right) \\ &= L, \end{aligned}$$

where the third line follows from $H|_{(V\times\{0\})\times(V\times\{0\})} = 0$, $\chi = 1$ on $X \times \{0\}$, and $H((0,\, l_1),\, (z,\, 0)) = \mathrm{Im}(l_1(\lambda))$. Thus $\mathscr{L}$ satisfies property (1). An easy check shows that $a_{(0,\, l_0)} = 1$, showing that property (2) is also satisfied. $\square$