1 1/19 - Brian Katz

1.1 Abelian Varieties and Complex Tori

Let $V = \mathbb{C}^g$ and $\Lambda \simeq \mathbb{Z}^{2g} \subset V$ be a full lattice, by which we mean that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$. Then we call $X = V/\Lambda$ a complex torus. Note that there is a natural morphism $\pi : V \to X$ which makes V the universal covering space for X.

Recall that if $\tilde{X} \xrightarrow{p} X$, with $\tilde{x}_0 \mapsto x_0$, is the universal cover for X, then there is a bijection $\pi_1(X, x_0) \leftrightarrow p^{-1}(x_0)$. A loop in the fundamental group (or really a class) is sent to the endpoint of the lift of the loop to the cover. Similarly, to a point in the fiber, associate the loop which is the image of (the unique homotopy class of) the loop between $\tilde{x}_0$ and the point. Using this understanding, we can see that, for X one of our tori, $\pi_1(X, 0) \simeq \Lambda$. Furthermore, since this fundamental group is abelian, $H_1(X, \mathbb{Z}) \simeq \Lambda$.

But there are other ways to encode this information. Let $\{e_1, \ldots, e_g\}$ be a basis for V over $\mathbb{C}$ and $\{\lambda_1, \ldots, \lambda_{2g}\}$ be a basis for Λ over $\mathbb{Z}$. Then we can write $\lambda_i = \lambda_{i,1}e_1 + \cdots + \lambda_{i,g}e_g$ for each i. From these coeffiecients, we can build a matrix, which we will call the period matrix of X (which depends on our choice of basis for V over $\mathbb{C}$):

$$\Pi = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{2g,1} \\ \vdots & & \vdots \\ \lambda_{1,g} & \cdots & \lambda_{2g,g} \end{pmatrix}.$$

1.2 Maps Between Tori

There are two obvious maps between tori: group homomorphisms and translations ($t_a : X \to X$ defined by $x \mapsto x + a$). It turns out that these are all of the maps.

Proposition 1.1. *1) Any holomorphic map between* 2 *tori,* $f : X \to Y$*, such that* $f(0) = 0$ *is a homomorphism.*

2) Any holomorphic map between 2 *tori,* $f : X \to Y$*, can be factored as* $f = t_{f(0)} \circ g$*, where* g *is a homomorphism.*

Proof. First note that 2) follows immediately from 1). Let $X = V/\Lambda$ and $Y = V'/\Lambda'$. Then we have the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\pi}\uparrow & \nearrow{\scriptstyle f\circ\pi} & \uparrow{\scriptstyle\pi'} \\ V & & V' \end{array}$$

But, from algebraic topology, by the contractibility of V we know that there is a unique lift of the diagonal map to a (holomorphic) map $F : V \to V'$ such that $F(0) = 0$ and $\pi' \circ F = f \circ \pi$. But then $F(z + \lambda) - F(z) \in \Lambda'$ for all $z \in V$ and $\lambda \in \Lambda$. By the discreteness of Λ', this implies that $F(z + \lambda) - F(z) = F(\lambda)$. This is not quite the desired periodicity. However, this does imply that $\frac{\partial F}{\partial z}$ is periodic and entire, hence constant. Thus F is linear. □

In this proof, we've constructed a map, $Hom(X, Y) \xhookrightarrow{\rho_a} Hom_{\mathbb{C}}(V, V')$. If f is a homomorphism between X and Y then we call $\rho_a(f) : V \to V'$ the analytic representation of f.

Similarly, these maps are all determined by their values on the lattice, so we have another map, $Hom(X,Y) \overset{\rho_r}{\hookrightarrow} Hom_{\mathbb{Z}}(\Lambda,\Lambda')$. We will call $\rho_r(f)$ the rational representation of f. Intuition tells us that there should be a simple relationship between these two maps, probably using the period matrices from before.

Picking bases for V, V', Λ, and Λ', we have that $\Pi \in M_{g,2g}(\mathbb{C})$, $\Pi' \in M_{g',2g'}(\mathbb{C})$, $\rho_a(f) = A \in M_{g',g}(\mathbb{Z})$ and $\rho_r(f) = R \in M_{2g',2g}(\mathbb{Z})$. Then these matrices satisfy the relation discovered by trying to make the matrix sizes match:

$$A\Pi = \Pi' R.$$

Proposition 1.2. *Let $f : X \to Y$ be a homomorphism of tori. Then $Im(f) \subset Y$ and $ker(f)_0 \subset X$ (the connected component of the kernel containing 0) are subtori.*

Proof. Note that $\rho_a(f) = F : V \to V'$. Then $Im(f) = F(V)/\Lambda \cap F(V)$. The question the becomes whether this is a full lattice. But Λ generates V over $\mathbb{R}$, so by linearity, $F(\Lambda)$ will generate $F(V)$ over $\mathbb{R}$. The proof is similar for $ker(f)_0$. □

Definition 1. *An **isogeny** is a finite surjective homomorphism between tori.*

Notation 2. *We will denote the multiplication map of degree n from X to itself by $[n]_X$. Also, we will denote $ker([n]_X) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$ by $X[n]$.*

Lemma 1.3. *Let $f : X \to Y$ be an isogeny of degree d. Then there exists an "inverse" isogeny $g : Y \to X$ such that $g \circ f = [d]_X$ and $f \circ g = [d]_Y$.*

Proof. By linearity $\#ker(f) = d$. A subgroup of order d is aniliated by d, so $ker(f) \subseteq X[d]$. But this means that multiplication by d determines the same element of X for any relresentative of a point in $X/ker(f)$. Furthermore, the first isomorphism theorem tells us that $X/ker(f) \simeq Y$ via the induced isomorphism $\overline{f}$.

$$\begin{array}{ccc} X/ker(f) & \xrightarrow{\overline{f}} & Y \\ {\scriptstyle [d]_X}\downarrow & \swarrow{\scriptstyle g} & \\ X & & \end{array}$$

Combining these two maps, we get a map $g = [d]_X \circ \overline{f}^{-1} : Y \to X$ which is clearly an isogeny such that $g \circ f = [d]_X$. Applying the same reasoning to g, there is an $h : X \to Y$ such that $h \circ g = [d]_Y$. But

$$[d]_Y \circ f = h \circ g \circ f = h \circ [d]_X = [d]_Y \circ h.$$

Then $[d]_Y \circ (f - h) = 0$, so $f - h = 0$ and we get the description of the other composition, $f \circ g = [d]_Y$. □

Corollary 1.4. *Isogeny is an equivalence relation on tori.*

The lemma proves the hard part of the corollary. And thus it makes sense to consider tori up to isogeny, which we will usually do.

1.3 Characteristic $p > 0$

Our simple description of homomorphisms is not true in finite characteristic, so there can be non-trivial isogenies here. Suppose X is an abelian variety over k, an algebraically closed field of finite characteristic p. Then there exists a purely inseparable isogeny $F : X \to X^{(p)}$ of degree p^g where $g = dim(X)$. Of course a few of these things are undefined, and the following discussion is for the students with a greater background in algebraic geometry.

If $U \subseteq X$ is an affine patch, then $U = Spec(A)$, where $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. Then $U^{(p)} \subset X^{(p)}$ will be a patch $Spec(A^{(p)}) = k[x_1, \ldots, x_n]/(f_1^{(p)}, \ldots, f_n^{(p)})$, where $f_i^{(p)}$ is the polynomial f_i with all of the coefficients raised to the p^{th} power. Note that $X^{(p)}$ is obtained from X by the base change $X^{(p)} = X \otimes_k k$, with a nontrivial map on k, $t \mapsto t^p$.

Then $F : X \to X^{(p)}$ is the identity on points, but on rings it is raising to the p^{th} power, an isogeny of degree p^g.

Homework 1.5. *What is the inverse in local coordinates? If you get stuck, look up Verschiebung.*

1.4 Embedding Our Tori

Thus far we have been considering complex tori. But an abelian variety is projective, so we need to embed them. This is done by the well known Weierstrass $\wp$-function in the one dimensional case.

Let $X = \mathbb{C}/\Lambda$; we want $X \to \mathbb{P}^n$ for some n. We'll try to use elliptic functions, by which we mean Λ periodic functions on $\mathbb{C}$.

Lemma 1.6. *Let f be (Λ) elliptic with P a fundamental domain.*

(1)

$$\sum_{z \in P} ord_z(f) = 0$$

(2)

$$\sum_{z \in P} z \cdot ord_z(p) \in \Lambda$$

Proof. For the first part, use the Residue Theorem for the function f'/f. For the second, use the Residue Theorem for zf'/f. □

Corollary 1.7. *A non constant elliptic function must have at least two poles in a fundamental domain.*

Proof. If it had only one pole, it would have only one zero by the first part of the theorem. But then, by the second part, the difference of those two points would lie on the lattice, which is impossible for geometric reasons. □

Question 1.8. *Can an elliptic function have just two poles?*

Yes, Weierstrass simply wrote down an elliptic function for any period lattice, Λ, with a single double pole in the fundamental region:

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus 0} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \\
&= \frac{1}{z^2} + \sum \frac{1}{\lambda^2}\left(\frac{1}{(1-z/\lambda)^2} - 1\right) \\
&= \frac{1}{z^2} + \sum \frac{1}{\lambda^2}\left[\left(1 + \frac{z}{\lambda} + \left(\frac{z}{\lambda}\right)^2 + \cdots\right)^2 - 1\right] \\
&= \frac{1}{z^2} + \left(\sum \frac{2}{\lambda^3}\right)z + \left(\sum \frac{3}{\lambda^4}\right)z^2 + \cdots \\
&= \frac{1}{z^2} + 3s_4 z^2 + 5s_6 z^4 + \cdots ,
\end{aligned}$$

where the s_i are the Eisenstein series. From this we can see that $\wp(z)$ is even and meromorphic. Also, we can compute, from the original definition, that $\wp'(z) = -2\sum \frac{1}{(z-\lambda)^3}$, which is obviously odd and periodic. Hence $\wp(z+\lambda) = \wp(z)+c$. If we use $z = -\lambda/2$ and evenness, we see that $c = 0$ and that $\wp(z)$ is periodic. Furthermore, it's not hard to construct a differential equation which $\wp(z)$ satisfies, namely:

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3.$$

The g_i are multiples of the s_j from above. We will use this function and its derivative to embed X in $\mathbb{P}^2$, sending z to $[1 : \wp(z) : \wp'(z)]$. Of course, this is not defined at 0. But since the functions are meromorphic, with poles of order 2 and 3 respectively at the origin, we can replace the embedding by $z \mapsto [z^3 : z^3\wp(z) : z^3\wp'(z)]$. Evaluated at 0, this is $[0 : 0 : 1] \in \mathbb{P}^2$. If we let the $\wp(z)$ coordinate be x and the $\wp'(z)$ coordinate be y, then the differential equation tells us the equation of the embedded curve is

$$y^2 = 4x^3 - g_2x - g_3.$$

So we've embedded X as a cubic.

What about adapting this to higher dimensions? Perhaps we could get a collection of meromorphic functions on X, $\{f_0(z), \ldots, f_N(z)\}$ and send $z \mapsto [f_0(z) : \cdots : f_N(z)]$. However, for functions of $g > 1$ variables, singularities are no longer points. There do exist meromorphic functions on higher dimensional varieties which cannot be extended to functions to projective space. We could fix this by requiring that our functions be holomorphic and periodic, but then Liouville's Theorem would force them to be constants. But we don't need the full power of periodicity. If we have that $f_i(z+\lambda) = F(z)f_i(z)$, independent of i, then in the projective space we will achieve periodicity. We will call this condition quasiperiodicity.