Algebraic Geometry

1 4/25/06

2 Trisecant lines

Theorem 2.1. *Sunppose* $\big(X, \Theta\big)$ *is a pricipally polarized abelian variety. Let* x, y, z *be points satisfying* $x \neq y \neq z, x \neq y + z$.

Then the following are equivalent:

1) $\Theta \cap \Theta_x \subset \Theta_y \cup \Theta_z$.

2) There exists constants $c_1, c_2, c_3 \in \mathbb{C} - \{0\}$ *such that:*

$$c_1\theta(\xi)\theta(\xi - y - z) + c_2\theta(\xi - y)\theta(\xi - z) + c_3\theta(\xi - x)\theta(\xi + x - y - z) = 0$$

3) $\psi\left(\frac{y+z}{2}\right), \psi\left(\frac{y-z}{2}\right), \psi\left(\frac{x-\frac{y+z}{2}}{2}\right)$ *are collinear.*

Recall earlier we had the following result due to Riemann:

$$H^0(X, \mathcal{O}(2\Theta)) = \mathbb{C}\Big\{\psi_x(\xi)\Big\}$$

$$w \in K(2\Theta)_1$$

$$\theta(z + a)\theta(z - a) = \Sigma_w \psi_x(a)\psi_w(z)$$

Proof of theorem:

Proof.
$2 \implies 1$ This is obvious.

$1 \implies 2$

$$\Theta \cap \Theta_x \subset \Theta_y \cup \Theta_z$$

$$\Leftrightarrow$$

$$\theta(\xi - y)\theta(\xi - z) = f(\xi)\theta(\xi) + g(\xi)\theta(\xi - x)$$

for some functions f and g.

Restrict to Θ. Then $f(\xi)\theta(\xi) = 0$. Then

$$g \in H^0(X, \mathcal{O}(\Theta_y + \Theta_z - \Theta_x|_\Theta)$$

$$= H^0(\mathcal{O}(\Theta_{y+z-x})|_\Theta)$$

There is a long exact sequence in cohomology in which a few of the terms are:

$$H^0(X, \mathcal{O}(\Theta_{y+z-x} - \Theta) \rightarrow H^0(X, \mathcal{O}(\Theta_{y+z-x})) \rightarrow H^0(X, \mathcal{O}(\Theta_{y+z-x})|_\Theta) \rightarrow H^1(X, \mathcal{O}(\Theta_{y+z-x} - \Theta))$$

Now a well known fact says that if $L \in \mathrm{Pic}^0(X)$ for some abelian variety X, and $X \neq 0$, then $H^i(X, L) = 0$. This says that

$$H^0(X, \mathcal{O}(\Theta_{y+z-x} - \Theta)) = H^1(X, \mathcal{O}(\Theta_{y+z-x} - \Theta)) = 0$$

Then

$$g(\xi) = c\theta(\xi + x - y - z) + \theta(\xi) + u(\xi)$$

for some constant c and some function u.

This gives

$$\theta(\xi - y)\theta(\xi - z) = f(\xi)\theta(\xi) + g(\xi)\theta(\xi - x)$$

$=$

$$c\theta(\xi + x - y - z)\theta(\xi - x) + \theta(\xi)v(\xi)$$

By the same type of argument

$$v(\xi) = \theta(\xi - y - z)$$

and we are done.

$2 \Leftrightarrow 3$

We have an embedding

$$\Psi : X \longrightarrow \mathbb{P}^{2^g - 1}$$

$$\Psi(x) = [\psi_w(x)]_{w=1}^{2^g}$$

Three points are colinear if and only if there are constants c_1, c_2, c_3 satisfying

$$0 = c_1\psi_w(\frac{y+z}{2}) + c_2\psi_w(\frac{y-z}{2}) + c_3\psi_w(x - \frac{y+z}{2}) \text{ for all } w.$$

from 2) perform the transformation

$$\xi \rightarrow \xi + \frac{y+z}{2}$$

We get

$$c_1\theta(\xi + \frac{y+z}{2})\theta(\xi - \frac{y+z}{2}) + c_2\theta(\xi - \frac{y-z}{2})\theta(\xi + \frac{y-z}{2}) + c_3\theta(\xi - x + \frac{y+z}{2})\theta(\xi + x - \frac{y+z}{2}) = 0$$

From Riemann's theorem we get

$$c_1 \sum_w \psi_w(\xi)\psi_w\Big(\frac{y+z}{2}\Big) + c_2 \sum_w \psi_w(\xi)\psi_w\Big(\frac{y-z}{2}\Big) + c_3 \sum_w \psi_w(\xi)\psi_w\Big(x - \frac{y+z}{2}\Big) = 0$$

and so

$$2 \Leftrightarrow 3$$

$\square$

Theorem 2.2. *Let C be a curve.*

$$X = Jac(C) \qquad\qquad C \hookrightarrow X$$

$x, y, z \in C$ be distinct points i.e $x \neq y \neq z \neq w \neq x \neq y$
Then

1) $\Theta \cap \Theta_{x-y}$ *is reducible.*

2) $\Theta \cap \Theta_{x-y} \subset \Theta_{x-z} \cup \Theta_{w-y}$

Remark 2.3. *This will give a four dimensional family of trisecant lines.*

Proof. $\Theta \subset \mathrm{Pic}^{g-1}(C)$ Let $L \in \Theta \cap \Theta_{x-y}$
Then we have

$$L \in \Theta \Rightarrow \mathrm{h}^0(L) \geq 1$$

$$L \in \Theta_{x-y} \Rightarrow \mathrm{h}^0(L \otimes \ \mathcal{O}_C(x-y)) \geq 1$$

$$\mathrm{h}^0(L \otimes \mathcal{O}_C(x)) = \begin{cases} 1 & \text{or} \\ 2 & \end{cases}$$

We have two cases:

1) $\mathrm{h}^0(L \otimes \mathcal{O}_C(y-z)) \geq 1$ for all z.
2) $\mathrm{h}^0(L \otimes \mathcal{O}_C(y-z)) = 1 \Leftrightarrow y$ is a base point for L.
   In this case
   $$\mathrm{H}^0(L \otimes \mathcal{O}_C(x)) = \mathrm{H}^0(L) = \mathrm{H}^0(L \otimes \mathcal{O}_C(-y)$$
   and we have
   $$\mathrm{H}^0(L \otimes \mathcal{O}_C(x)) = \mathrm{H}^0(L \otimes \mathcal{O}_C(x-y).$$
   So
   $$\mathrm{h}^0(L \otimes \mathcal{O}_C(w-y)) \geq 1.$$

We then have

$$L \in \Theta_{x-z} \text{ or } L \in \Theta_{w-y}.$$

and

$$\Theta \cap \Theta_{x-y} = \big\{L : \mathrm{h}^0(L \otimes \mathcal{O}_C(\Theta_x)) \geq 2)\big\} \bigcup \big\{L : \mathrm{h}^0(L \otimes \mathcal{O}_C(\Theta_{-y})) \geq 1)\big\}$$

Recall

$$W_d = \big\{E \in \mathrm{Pic}^d(C) : \mathrm{h}^0(E) \geq 1\big\}$$

Lets apply Riemann - Roch to figure out what

$$\big\{L : \mathrm{h}^0(L \otimes \mathcal{O}_C(\Theta_x)) \geq 2\big\}$$

is.

Riemann-Roch says that

$$\mathrm{h}^0(L + x) - h^0(k_C - L - x) = g + 1 - g = 1$$

so

$$\big\{L : \mathrm{h}^0(L \otimes \mathcal{O}_C(\Theta_x)) \geq 2\big\} = \big\{L : \mathrm{h}^0(k_C - L - x) \geq 1\big\}$$

$$2g - 2 - g + 1 - 1 = g - 2$$

so that

$$k_C - L - x \in W_{g-2} \text{ or } L \in k_C - x - W_{g-2}.$$

Therefore the components of $\Theta \cap \Theta_{x-y} = y + W_{g-2} \cup (k_C - x - W_{g-2})$

□

We have lots of trisecants.

Remark 2.4. *For a generic* (X, Θ)

$$\Theta \textit{ is smooth}$$

$$\Theta \cap \Theta_x \textit{ is smooth and irreducible}$$

$$\Theta \subset \Theta_{x-y} \cup \Theta_{w-y}$$

Trisecants.

$$\Psi\Big(\frac{x + w - y - z}{2}\Big), \Psi\Big(\frac{x + y - w - z}{2}\Big), \Psi\Big(\frac{x - y - w + z}{2}\Big)$$

are collinear. For fixed $y, z, w \in C$ and $\xi = \frac{1}{2}(c - y - z - w)$

$$\Psi(\xi + y), \Psi(\xi + z), \Psi(\xi + w)$$

are collinear. This parameterizes the tri-secants.

Question: Does this characterize Jacobians?

Some hand waving...

Answer: Yes and No.

Theorem 2.5. *(Gunning)*

If (X, Θ) *is an abelian variety such that there is a curve* $C \subset X$ *a curve with the property that* $y, z, w \in C \quad \xi \in (C - y - z - w)$*, and the points*

$$\Psi(\xi + y), \Psi(\xi + z), \Psi(\xi + w)$$

are collinear, then

$$X = Jac(C)$$

and C *is the image of the Abel-Jacobi map.*