1 1/31 - Paul Larsen

1.1 Abelian Varieties and Complex Tori

Notices: The course webpage is now available off of Dr. Farkas' webpage. We will soon be discussing course projects, which are to be done by the end of the semester.

1.2 Recap

Let X be a complex manifold (for our purposes, thinking $X = \mathbb{C}^g$ will usually suffice). If $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ is a morphism of sheaves, then for any open $U \subset X$, define

$$ker(\phi)(U) = ker(\phi|_U). \tag{1}$$

The naïve definition for the image of a morphism, however, does not yield a sheaf, as in the example from last time involving evaluation maps. Instead we define

$$\begin{gathered} Im\ \phi(U) = \{s \in \mathcal{G}(U) : \forall p \in U, \exists V \subset U \text{ with } p \in V, \text{ and} \\ \exists t \in \mathcal{F}(V) \text{ s.t. } \phi_V(t) = s|_V\}. \end{gathered}$$

1.3 Exact Sequences of Sheaves

Let

$$0 \to \mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{j} \mathcal{F}'' \to 0 \tag{2}$$

be an exact sequence of sheaves, that is, i is injective, and j is surjective (where surjectivity means $Im(j) = \mathcal{F}''$, not that $j(U)$ is surjective for all $U \subset X$). In terms of the evaluation map example from last time, for $p, q \in X$, $\mathcal{O}_x \xrightarrow{\text{eval.}} \mathbb{C}_p \oplus \mathbb{C}_q$ is surjective as a sheaf morphism, but not globally (recall that X is assumed to be compact, so the only global sections are constant functions). Restricting to the open sets $\mathbb{C} - \{p\}$ and $\mathbb{C} - \{q\}$ in turn, however, the morphism is surjective. The global failure of this example is what brings us to consider cohomology.

1.4 Sheaf Coholomogy via Čech cohomology

From the failure of the evaluation morphism to be surjective, we see that for a general exact sequence of sheaves, the last map of (2) will not be to zero. Instead, it will be of the form

$$0 \to H^0(\mathcal{F}') \xrightarrow{i} H^0(\mathcal{F}) \xrightarrow{j} H^0(\mathcal{F}'') \to H^1(\mathcal{F}') \to H^1(\mathcal{F}) \to H^1(\mathcal{F}'') \to H^2(\mathcal{F}') \to \dots \tag{3}$$

To any short exact sequence of sheaves, we will associate a long exact sequence in cohomology, which will be functorial. There are many ways to define sheaf coholomogy; we will use Čech cohomology.

Definition 1 (Čech cochain complex). *Let X be a topological space, $\mathcal{F}$ be a sheaf on X, and $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover, with Λ a totally ordered index set (which is fine, assuming the reader has no qualms with the axiom of choice)*

Define the cochain complex, *$C^\bullet(\mathcal{U}, \mathcal{F})$ by*

$$C^p(\mathcal{U}, \mathcal{F}) = \Pi_{\alpha_0 < \ldots < \alpha_p} \mathcal{F}(\mathcal{U}_{\alpha_0 \ldots \alpha_p}), \tag{4}$$

for $p \geq 0$, and where $U_{\alpha_0 \ldots \alpha_p} = U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}$.

For example, $C^0(\mathcal{U}, \mathcal{F}) = \Pi_\alpha \mathcal{F}(U_\alpha)$, i.e., the (ordered) collection of sections s_α over U_α, written $s = \{s_\alpha \in \mathcal{F}(U_\alpha)\}$ or $s = (s_\alpha)_{\alpha \in \Lambda}$.

We now define the coboundary map for a Čech cochain complex.

Definition 2 (Coboundary map). *For $p \geq 0$, define the* coboundary map $d^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ *for $\sigma \in C^p(\mathcal{U}, \mathcal{F})$, $\sigma = \{\sigma_{\alpha_0 \ldots \alpha_p}\}$ by*

$$d^p(\sigma)_{\beta_0 \ldots \beta_{p+1}} = \sum_{j=0}^{p+1} \sigma_{\beta_0 \ldots \hat{\beta}_j \ldots \beta_{p+1}}|_{U\beta_0 \ldots \beta_{p+1}} \tag{5}$$

For example, if $\mathcal{U} = \{U_\alpha, U_\beta\}$, and $\sigma \in C^0(\mathcal{U}, \mathcal{F})$, then

$$d(\sigma)_{\alpha\beta} = \sigma_\beta|_{U\alpha \cap U\beta} - \sigma_\alpha|_{U\alpha \cap U\beta}. \tag{6}$$

Thus as long as $d^{p+1} \circ d^p = 0$ (which is just a combinatorial calculation), we have a cochain complex

$$\ldots \overset{d^{p-1}}{\to} C^p(\mathcal{U}, \mathcal{F}) \overset{d^p}{\to} C^{p+1}(\mathcal{U}, \mathcal{F}) \overset{d^{p+1}}{\to} C^{p+2}(\mathcal{U}, \mathcal{F}) \overset{d^{p+2}}{\to} \ldots \tag{7}$$

Definition 3 (Čech cohomology groups relative cover). *We define the p^{th}* Čech cohomology group *relative the covering $\mathcal{U}$ as*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \frac{ker(d^p)}{im(d^{p-1})} \tag{8}$$

This definition relies on a choice of open cover for X, but we want to frame cohomology independently of this choice. We may consider all covers and take the inductive limit, but if we pick a suitably refined cover, the situation is not as bad as it may appear.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in I'}$ be two coverings of X. We define an ordering, $\mathcal{V} < \mathcal{U}$, on these coverings, and say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if for every $\beta \in I'$, there exists an $\alpha \in I$ such that $V_\beta \subseteq U_\alpha$. Hence we may choose a map $\phi : I' \to I$ such that for all $\beta \in I'$, $V_\beta \subseteq U_{\phi(\beta)}$, thus giving a map on $p-$cochains

$$\rho_\phi : C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F}) \tag{9}$$

given for $\sigma \in C_p(\mathcal{U}, \mathcal{F})$ by,

$$\rho_\phi(\sigma)_{\beta_0 \ldots \beta_p} = \sigma_{\phi(\beta_0) \ldots \phi(\beta_p)}|_{V\beta_0 \ldots \beta_p}. \tag{10}$$

It is easy to check that $\rho_\phi \circ d^p = d^p \circ \rho_\phi$, so ρ_ϕ induces a homomorphism on cohomology

$$\rho : \check{H}^p(\mathcal{U},\mathcal{F}) \to \check{H}^p(\mathcal{V},\mathcal{F}). \tag{11}$$

Likewise, it follows from the construction above and the sheaf axioms that for any other choice of maps $\psi : I' \to I$, both ϕ and ψ induce the same map on cohomology. We may define Čech cohomology independently of the choice of open covering.

Definition 4 (Čech cohomology groups).

$$\check{H}^p(X,\mathcal{F}) = \lim_{\overrightarrow{\mathcal{U}}} H^p(\mathcal{U},\mathcal{F}).$$

As mentioned before, the situation is not as bad as the above may intimate, thanks to the following assumed

Proposition 1.1. *If the covering $\mathcal{U}$ is 'fine enough,' then*

$$\check{H}^p(X,\mathcal{F}) \cong H^p(\mathcal{U},\mathcal{F}). \tag{12}$$

We will not spend time on what 'fine enough' means, but rather will consider a few examples. First, we require an additional definition.

Definition 5 (Coherent sheaf). *Let $\mathcal{O}_X$ be the sheaf of holomorphic functions, then a sheaf $\mathcal{F}$ is said to be* coherent *if $\mathcal{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$ module for all open $U \subset X$.*

Note that this definition is equivalent to saying that the sequence

$$\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0$$

is exact, since if we pick a set of n generators for $\mathcal{F}$, then $\mathcal{F}$ is isomorphic to the cokernel of some map from $\mathcal{O}_X^m$ to $\mathcal{O}_X^n$ that gives the relations in $\mathcal{F}$.

One example of when Proposition 1.1 holds is if X is a complex projective variety, $\mathcal{F}$ is a coherent sheaf, and $\mathcal{U}$ is an open covering of affine sets. In the topological category, if $\mathcal{F}$ is locally constant, then the proposition holds if all $U_{\alpha_0\cdots\alpha_p}$ are locally contractible.

For our purposes, we may generally ignore the inductive limit definition and work with a cover that will be 'suitably fine.' We will use this assumption in the following example/proposition.

Proposition 1.2. *Let X be a topological space, $\mathcal{F}$ a sheaf over X, then $\check{H}^0(X,\mathcal{F}) = \mathcal{F}(X)$.*

Proof. Pick a covering $\mathcal{U} = \{U_\alpha\}$ such that $\check{H}^0(X,\mathcal{F}) = \check{H}^0(\mathcal{U},\mathcal{F})$. Then if $\sigma \in \check{H}^0(\mathcal{U},\mathcal{F})$, $\sigma = \{\sigma_\alpha \in \mathcal{F}(U_\alpha)\}$. By the sheaf axioms, for all α,β, $\sigma_\alpha|_{U\alpha\beta} = \sigma_\beta|_{U\alpha\beta}$, hence there exists a unique $\tilde{\sigma} \in \mathcal{F}(X)$ such that $\tilde{\sigma}|_{U\alpha} = \sigma_\alpha$. ☐

1.5 Long Exact Sequences in Cohomology

(From here forward, we will write $H^\bullet(\mathcal{F})$ rather than $\check{H}^\bullet(\mathcal{F})$ when the context is clear.)

Note that Čech cohomology is functorial, as a morphism $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ induces a homomorphism $H^p(\mathcal{F}) \to H^p(\mathcal{G})$ via the induced map on the chain level.

Let

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be a short exact sequence of sheaves. Then there is an associated long exact sequence in homology,

$$H^0(\mathcal{F}') \to H^0(\mathcal{F}) \to H^0(\mathcal{F}'') \xrightarrow{\delta} H^1(\mathcal{F}') \to H^1(\mathcal{F}) \to H^1(\mathcal{F}'') \xrightarrow{\delta} H^2(\mathcal{F}') \to \dots \tag{13}$$

We can define the coboundary maps δ by taking a sufficiently fine open over $\mathcal{U} = \{U_\alpha\}$ of X, and using the standard construction involving diagram chasing (e.g., in Hatcher, *Algebraic Topology*, p. 116-117, or Griffiths and Harris, *Principles of Algebraic Geometry*, p. 31).

Question 1.3. *Is it obvious that the (induced diagram on chains for the short exact sequence above) is commutative?*

Yes, but you have to do it once for yourself. The argument uses functoriality and the sheaf axioms.

Next, we will return to Dolbeault cohomology.

1.6 Dolbeault cohomology

Let X be a complex manifold (you may safely think of $\mathbb{C}^g$). We have sheafs $\mathcal{A}_X^n$,the C^∞ $n-$forms on X, and $\mathcal{A}_X^{a,b}$, the C^∞ (a,b) forms on X, and we have the $\bar{\partial}$ operator that gives the Dolbeault cohomology groups $H^{(a,b)}(X) = \frac{ker(\bar{\partial}_{(a,b)})}{im(\bar{\partial}_{(a,b-1)})}$ as before.

Theorem 1.4. *Let X be a complex manifold. Then $H^p(X, \mathcal{A}_X^n) = H^p(X, \mathcal{A}_X^{(a,b)}) = 0$ for all $p > 0$.*

So these sheaves have no interesting cohomology and we will use them mainly as a technical tool in what follows.

Proof. We only prove that $H^1(X, \mathcal{A}_X^n) = 0$, the general case being entirely similar.. Choose a sufficiently fine open cover $\mathcal{U} = \{U_\alpha\}$ of X. Then for $\sigma \in H^1(X, \mathcal{A}_X^n)$, $\sigma = \{\sigma_{\alpha\beta}\}_{\alpha<\beta}$, we have that $d^1\sigma = 0$ if and only if for all $\alpha < \beta < \gamma$,

$$(\sigma_{\beta\gamma} - \sigma_{\alpha\gamma} + \sigma_{\alpha\beta})|_{U\alpha\beta\gamma} = 0. \tag{14}$$

We produce a $0-$cycle τ whose differential is σ. For this we use partitions of unity (which exist on X as it is a manifold, and hence a paracompact space).

Pick a partition of unity $\{\rho_\alpha\}$ subordinate to $\mathcal{U}$, i.e., $\rho_\alpha \in C^\infty(X)$ are such that $\text{supp}(\rho_\alpha) \subseteq U_\alpha$ for all α and $\sum_\alpha \rho_\alpha = 1$. Define $\tau \in C^0(\mathcal{U}, \mathcal{A}_X^n)$, $\tau = \{\tau_\alpha\}$ by

$$\tau_\alpha = -\sum_\gamma \rho_\gamma \sigma_{\alpha\gamma}. \tag{15}$$

We verify now that $d^0(\tau) = \sigma$. Note that we have extended the $\sigma_{\alpha\gamma}$ by zero to all of X by multiplying by ρ_γ. Hence we need not worry about domains of definition in the following calculation.

$$\begin{aligned}(d\tau)_{\alpha\beta} &= \tau_\beta - \tau_\alpha \\ &= \sum_\gamma \rho_\gamma(\sigma_{\alpha\gamma} - \sigma_{\beta\gamma}) \\ &= \big(\sum_\gamma \rho_\gamma\big)(\sigma_{\alpha\beta}) \\ &= \sigma_{\alpha\beta},\end{aligned}$$

where the second to last equality holds since σ is a cocycle (so relation (14) holds). Hence $\sigma = 0$ in $H^1(X, \mathcal{A}_X^n)$, and so $H^1(X, \mathcal{A}_X^n) = 0$. □

The above will prove useful in computing Dolbeault coholomogy, and in particular in the following theorem.

Theorem 1.5 (Dolbeault's Theorem). *Let X be a complex manifold. Then*

$$H^{p,q}_{Dol}(X) \cong H^q(X, \Omega_X^p) \tag{16}$$

(where the second is the $q-$th Čech cohomology of Ω_X^p, the sheaf of holomorphic $p-$forms on X).

Instead of proving Dolbeault's theorem, we will prove a baby version whose proof is formally the same as for Dolbeault's theorem.

Theorem 1.6 (deRham's Theorem). *Let X be a complex manifold. Then*

$$H^p(X, \mathbb{C}) \cong H^p_{dR}(X),$$

where the former is taken to be either the singular cohomology of X or the Čech cohomology of X over the constant sheaf.

Proof. We will make use of Poicaré's lemma, i.e., that the following sequence of sheaves is exact:

$$0 \to \mathbb{C} \to \mathcal{A}_X^0 \xrightarrow{d^0} \mathcal{A}_X^1 \xrightarrow{d^1} \mathcal{A}_X^2 \xrightarrow{d^2} \ldots \tag{17}$$

The above is a local statement: exactness at $\mathcal{A}_X^0$ follows by observing that if the derivative a function is zero, it is a constant function. Exactness at $\mathcal{A}_X^1$ follows from the (local) existence of antiderivatives.

We can break the sequence (17) into a series of short exact sequences:

$$0 \to \mathbb{C} \to \mathcal{A}_X^0 \to ker(d^1) \to 0, \tag{18}$$

$$0 \to ker(d^1) \to \mathcal{A}_X^1 \to ker(d^2) \to 0, \ldots \tag{19}$$

$$\vdots$$

$$0 \to ker(d^{p-1}) \to \mathcal{A}_X^{p-1} \to ker(d^p) \to 0, \ldots \tag{20}$$

Each of the above short exact sequences gives a long exact sequence in cohomology. In particular, from the first short exact sequence, we have the segment

$$\ldots \to H^{p-1}(X, \mathcal{A}_X^0) \to H^{p-1}(ker(d^1)) \to H^p(X, \mathbb{C}) \to H^p(X, \mathcal{A}_X^0) \to \ldots, \tag{21}$$

but the first and last elements are zero by theorem (1.4), hence $H^{p-1}(X, ker(d^1)) \cong H^p(X, \mathbb{C})$. Likewise, the second short exact sequence yields $H^{p-1}(X, ker(d^1)) \cong H^{p-2}(X, ker(d^2))$. Iterating, we have $H^p(X, \mathbb{C}) \cong H^1(X, ker(d^{p-1}))$.

This isomorphism plus the final short exact sequence, (20), give

$$\ldots \to H^0(X, \mathcal{A}_X^{p-1}) \to H^0(ker(d^p)) \to H^p(X, \mathbb{C}) \to H^1(X, \mathcal{A}_X^{p-1}) \to \ldots.$$

Again, the final term listed above is zero, thus we have

$$H^p(X, \mathbb{C}) \cong \frac{H^0(ker(d^p))}{Im\big(H^0(\mathcal{A}_X^{p-1})\big)} \cong H^p_{dR}(X), \tag{22}$$

as desired. □

For Dolbeault cohomology, we will need a $\bar{\partial}$-Poincaré lemma to obtain a resolution of the sheaf of holomorphic forms, Ω_X^p, i.e., a long exact sequence of sheaves,

$$0 \to \Omega_X^p \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \xrightarrow{\bar{\partial}} \ldots. \tag{23}$$