1 Abelian Varieties 2/23 - Notes taken by Parker Lowrey

1.1 Setup

Recall that when $X = V/\Lambda$ was a complex torus, its dual was defined as $\hat{X} = \overline{\Omega}/\hat{\Lambda}$ where Λ is a full lattice of the complex vector space V and $\hat{\Lambda} = \{l \in \overline{\Omega} : \text{Im } l(\Lambda) \subseteq \mathbb{Z}\}$ is a full lattice in $\Omega = \text{Hom}_{\mathbb{C}-anti}(V, \mathbb{C})$. Then we have $\hat{X} \backsimeq \text{Hom}(\Lambda, S^1)$ obtained by the map $l \to e^{2\pi i \text{ Im } l(\lambda)}$ where $\lambda \in \Lambda$. With the above definitions then $X \times \hat{X} = V \times \overline{\Omega}/\Lambda \times \hat{\Lambda}$ becomes a complex torus.

1.2 Poincaré Bundle

By definition, a Poincaré bundle for X is a line bundle $\mathscr{L}$ over $X \times \hat{X}$ with the following two properties:

1. $\forall\ L \in Pic^0(X)$, $j_L^*(\mathscr{L}) = L$ where $j_L : X \hookrightarrow X \times \hat{X}$ is defined by $x \to (x, L)$
2. $\mathscr{L}\big|_{\{0\}\times\hat{X}} = \mathscr{O}_{\hat{X}}(triv)$

Propostion 1. *There exists a unique $\mathscr{L}$ on $X \times \hat{X}$ satisfying the above conditions.*

Proof. We first recall the following fact: If X and T are complex varieties (works in algebraic category as well) and $\mathscr{L}$ is a line bundle on the product then

1. $\exists$ a unique closed set $T_0 \subseteq T$ such that $\mathscr{L}_{X\times\{t\}}$ is trivial $\forall t \in T_0$ and T_0 is maximal with this property.
2. $\exists$ a line bundle $\mathscr{E}$ on T such that $\mathscr{L}\big|_{X\times T_0} = \pi^*\mathscr{E}$.

Now assume that $\exists\ \mathscr{L}, \mathscr{M}$ satisfying the above conditions. Then $\mathscr{N} = \mathscr{L} \otimes \check{\mathscr{M}}$ is a line bundle on $X \times \hat{X}$ such that $\forall L \in \text{Pic}^0(X)$, $j_L^*(\mathscr{N}) = \mathscr{O}_X$. Letting $T = \hat{X}$ and applying the above fact to our situation we have $T_0 = \hat{X}$ and that $\mathscr{N} = \pi_2^*\mathscr{E}$. However, $\mathscr{E} = \mathscr{N}\big|_{0\times\hat{X}} = \mathscr{O}_{\hat{X}}$ i.e. $\mathscr{E}$ is the trivial bundle on $\hat{X}$. Since the pull-back bundle of the trivial bundle is trivial we have $\mathscr{L} \simeq \check{\mathscr{M}}$ ☐

Definition 1. The unique line bundle $\mathscr{L}$ satisfying the two conditions above is call the Poincaré bundle on $X \times \hat{X}$

1.3 Maps between X and $\hat{X}$

Now let $\mathscr{L}$ be a line bundle on X. We have a map $\phi_L : X \to \hat{X}$ given by $x \to t_x^*(\mathscr{L}) \otimes \check{\mathscr{L}}$. As shown in past lectures, $t_x^* L(H, \chi) = L(H, \chi e^{2\pi i E(x,\bullet)})$, we can therefore write this map as $z \to H(z, \bullet)$ from $V \to \overline{\Omega}$, where $\mathscr{L} = L(H, \chi)$ (See notes from 2/16).

Propostion 2. *If $\mathscr{L}$ is a non-degenerate line bundle on X, then $c_1(\acute{\mathscr{L}}) = c_1(\mathscr{L})$ if and only if $\exists\ x \in X$ such that $\acute{\mathscr{L}} = t_x^* \mathscr{L}$.*

Proof. If $\mathscr{L}$ is non-degenerate the $\phi_{\mathscr{L}}$ is an isogeny. $\acute{\mathscr{L}} \otimes \check{\mathscr{L}} \in \hat{X} \Rightarrow \exists x \in X$ such that $\phi_{\mathscr{L}}(x) = \acute{\mathscr{L}} \otimes \check{\mathscr{L}} \Rightarrow t_x^* \mathscr{L} = \acute{\mathscr{L}}$. □

1.4 Theta Functions

Let X be as above, and L a line bundle on X with $L = L(H, \chi)$. We write the factors of automorphy obtained from H and χ (See notes from 2/9) as $a_\lambda(z) \in \mathrm{H}^0(\mathscr{O}_V^*)$. Recall that these are obtained through the following:

$$\begin{array}{ccc} L(\pi(z)) & \xrightarrow{\chi_z} & \mathbb{C} \\ \Vert & & \downarrow a_{\lambda(z)} \\ L(\pi(z+\lambda)) & \xrightarrow[\chi_{z+\pi}]{} & \mathbb{C} \end{array}$$

We would like to understand $\mathrm{H}^0(X, L) = \Gamma(L)$.

Now, let $s \in \mathrm{H}^0(X, L)$. Then $s : X \to L$ such that $s(z) \in L(z)$. We can lift this section to a map on V, and through χ, to a function ϑ on V:

$$\begin{array}{ccc} z & \longrightarrow & s(\pi(z)) \\ & \searrow^{\vartheta} & \downarrow \chi_z \\ & & \mathbb{C}. \end{array}$$

Thus we can describe $\mathrm{H}^0(X, L) = \{\vartheta \in \mathrm{H}(\mathscr{O}_V) : \vartheta(z + \lambda) = a_\lambda(z)\vartheta(z)\}$ This space is called the canonical theta functions for L.

Although easy to define, using $a_\lambda(z)$ is not the best to work with. Recall that if $\{e_\lambda\} \sim \{a_\lambda\}$, then $e_\lambda(z) = f(z+\lambda)a_\lambda(z)/f(z)$ for some $f \in H^0(\mathscr{O}_V^*)$. Thus we see that

$$\{\vartheta \in \mathrm{H}(\mathscr{O}_V) : \vartheta(z+\lambda) = a_\lambda(z)\vartheta(z)\} \longleftrightarrow \{\vartheta \in \mathrm{H}(\mathscr{O}_V) : \vartheta(z+\lambda) = e_\lambda(z)\vartheta(z)\}$$

$$\vartheta(z) \to \vartheta(z) f(z)$$

Using this equivalence, we would like to define a new factor of automorphy that will make our theta functions periodic. To do this we need to do some setup.

1.5 Isotropic decomposition for L

Let L and X be as above. $H = c_1(L)$ and $E = ImH$. Since E is an alternating form on Λ. there exists a unique basis $(\lambda_1, \dots, \lambda_g, \nu_1, \dots, \nu_g)$ of Λ such that the matrix of E with respect to this basis is

$$\begin{pmatrix} 0 & diag(d_1, \dots, d_g) \\ -diag(d_1, \dots, d_g) & 0 \end{pmatrix} \qquad \text{with} \qquad d_1 | d_2 | \dots | d_g \quad (1)$$

Note: L is non-degenerate $\iff d_i > 0 \; \forall \; i$, further $(d_1, \dots, d_g)$ is called the type of L.

Now, we are going to decompose our lattice and our vector space according to this basis: $\Lambda_1 = \mathbb{Z} < \lambda_1, \dots, \lambda_g >$, $\Lambda_2 = \mathbb{Z} < \nu_1, \dots, \nu_g >$ and $V_1 = \Lambda_1 \otimes_{\mathbb{Z}} \mathbb{R}$, $V_2 = \Lambda_2 \otimes_{\mathbb{Z}} \mathbb{R}$. Then $V = V_1 \oplus V_2$ and V_1 and V_2 are maximal isotropic spaces for E.

Propostion 3. *Choose a decomposition of $V = V_1 \oplus V_2$ into isotropic subspaces for L (or more precisely E). Then define $\chi_0 : \Lambda \to S^1$ by $\chi_0(\lambda) = e^{\pi i E(\lambda_1, \lambda_2)}$ where $\lambda = \lambda_1 + \lambda_2, \; \lambda_i \in \Lambda_i$. Then*

1. χ_0 is a semi-character defining $L_0 = L(H, \chi_0)$ i.e. if we have a cohomology class and a decomposition then we have a cannonical bundle.

2. L_0 is the only line bundle on X with $c_1(L) = H$ such that $\chi\big|_{\Lambda_1}$ and $\chi\big|_{\Lambda 2}$ are trivial.

3. Further assume H is non-degenerate, then $\forall L \; \in Pic(X)$ with $c_1(L) = H$, $\exists \; c \; \in V$ such that $L = t_c^* L_0$, c is called the characteristic of L.

Proof. By definition, $L_0 = L(H, \chi_0) \in Pic^H(X) = \{L \in Pic(X) \big| c_1(L) = H\}$,

1. This is trivial

2. Since Λ_1 and Λ_2 are isotropic subspaces for E, then $\chi\big|_{\Lambda_1} = 1$ and $\chi\big|_{\Lambda 2} = 1$

3. This follows from an earlier theorem, further the kernel of $\phi_L : X \to \hat{X}$ is $\frac{\Lambda(L)}{\Lambda}$ where $\Lambda(L) = \{z \in V \big| \text{Im}H(z, \Lambda) \subseteq \mathbb{Z}\}$. Thus $t_c^* L_0 = t_{c'}^* L_0 \Rightarrow c - c' \in \Lambda(L)$ and c is unique up to translation by an element in $\Lambda(L)$.

□

Now, we have $\vartheta(z + \lambda) = a_\lambda(z)\vartheta(z)$. We now want to alter a_λ to a homologous factor of automorphy e_λ which is periodic when restricted to Λ_1 and Λ_2. In fact, we will pick $e_\lambda = 1, \lambda \in \Lambda_2$.

Start with $H \in NS(X)$ and a decomposition $V = V_1 \oplus V_2$. Let B be the restriction of H to V_2. B is symmetric and is real valued since $H\big|_{V_2 \times V_2}(V_2 \times V_2) \subseteq \mathbb{R}$. Using the equivalence $V_2 \otimes_{\mathbb{R}} \mathbb{C} \cong V$ (which is due to $V_2 \subset V$ and the fact that V is a complex vector space) we extend B complex linearly to V Thus B is a complex valued symmetric $\mathbb{C}$-linear form on V.

Let $v, w \in V$ with $v = v_2 + iv_2', w = w_2 + iw_2'$. We have the following relations between H and B:

$$H(v, w) = H(v_2, w_2) + H(\acute{v_2}, \acute{w_2}) + i(H(\acute{v_2}, w_2) - H(v_2, \acute{w_2}))$$
$$B(v, w) = (H(v_2, w_2) - H(\acute{v_2}, \acute{w_2})) + i(H(\acute{v_2}, w_2) + H(v_2, \acute{w_2}))$$

$$\Rightarrow (H - B)(v, w) = 2H(\acute{v_2}, \acute{w_2}) - 2iH(v_2, \acute{w_2})$$

In order to aid future calculations we need the following two formulas

$$(H - B)\big|_{V_2 \times V} = 0 \qquad (H - B)\big|_{V_2 \times V} = 2iE$$

The first is clear from the definitions, the second can be calculated using

$$(H - B)\big|_{V_2 \times V}(v_2, w) = -2iH(v_2, \acute{w_2}) = -2iE(iv_2, \acute{w_2}) + 2E(V_2, \acute{w_2}) \quad (2)$$

and $H(x, y) = E(ix, y) + iE(x, y)$, for then

$$(H - B)\big|_{V_2 \times V}(v_2, w) = -2iE(iv_2, i\acute{w_2}) = 2iE(v_2, i\acute{w_2}) = 2iE(v_2, w) \quad (3)$$

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Definition 2. Classical factor of automorphy: $e_\lambda(z) = \chi(\lambda) e^{\pi(H-B)(z,\lambda) + \frac{\pi}{2}(H-B)(\lambda,\lambda)}$

It is easy to see that $\{e_\lambda\} \sim \{a_\lambda\}$ since $a_\lambda(z)/e_\lambda(z) = f(z + \lambda)/f(z)$ where $f(z) = e^{\frac{\pi}{2}B(z,z)}$. Thus

$$\{\text{classical theta functions}\} \longleftrightarrow \{\text{cannonical theta functions}\}$$
$$\vartheta(z) \longmapsto e^{\frac{\pi}{2}B(z,z)}\vartheta(z) \quad (4)$$

Note that classical theta functions are periodic with respect to Λ_2 ($c = 0$):

$$\lambda \in \Lambda_2 \Longrightarrow e_\lambda(z) = 1 \Longrightarrow \vartheta(z + \lambda) = \vartheta(z) \quad (5)$$

Lastly, the following will be expanded on next class.

Lemma 1. Let $f(z_1, \ldots, z_g) \in H^0(\mathscr{O}_V)$ with $f(z_1 + c_1, \ldots, z_g + c_g) = f(z_1, \ldots, z_g)$ then f has fourier expansion: $f(z_1, \ldots, z_g) = \sum_{n=(n_1,\ldots,n_g)\in\mathbb{Z}^g} d_n e^{2\pi i(\frac{z_1 n_1}{c_1} + \ldots \frac{z_g n_g}{c_g})}$

Theorem 1. If L is a positive definite line bundle on X having type $(d_1, \ldots, d_g)$ then $\dim H^0(X, L) = d_1 d_2 \ldots d_g$.