ABELIAN VARIETIES 3/23/06- NOTES TAKEN BY MARK LUXTON

1. More on Line Bundles on a Complex Torus

Theorem 1.1. *Let X be a complex torus, L a line bundle on X. Then the following are equivalent:*

(1) *L is ample.*
(2) *L is positive definite.*
(3) *$H^0(X, L) \neq 0$ and $\#k(L) < \infty$.*

In the course of proving this theorem, we will also prove the following:

Corollary 1.2. *If L is a line bundle on X and if $H^0(X, L) \neq 0$, L is positive semi-definite.*

Proof. We've seen that (2) $\Rightarrow$ (3); that (3) $\Rightarrow$ (2) was proven last time and (2) $\Rightarrow$ (1) is Lefschetz's theorem. It remains to show (1) $\Rightarrow$ (2).

To show (1) $\Rightarrow$ (2), we need only show that L is non-degenerate. Then by a theorem proved last time, L is ample, non-degenerate and so L is positive definite.

So suppose L is ample and let $H = c_1(L)$. To prove non-degeneracy we may assume that L is very ample since H is degenerate if and only if nH is degenerate.

Assume H is degenerate and let $N = \{x \in V \mid H(x, y) = 0 \text{ for all y} \in Y\}$ be the null-space of H. Then $N \neq 0$ is a subspace of V and $N \cap \Lambda$ is a sub-lattice of Λ.

Since L is very ample, $H^0(X, L) \neq 0$ so let $\vartheta \in H^0(X, L) \setminus \{0\}$ be a theta function. As before, let $\Bbbk \subset V$ be compact with $V = \Bbbk + \Lambda$. For any $z \in V$ we can then write $z = d + \lambda$ with $d \in \Bbbk, \lambda \in \Lambda$. Finally, fix some $z_0 \in V$ and let $z \in N$ be arbitrary.

Then

$$\vartheta(z_0 + z) = \vartheta(z_0 + d + \lambda) = a_\lambda(z_0 + d)\vartheta(z_0 + d).$$

But

$$a_\lambda(z_0 + d) = \chi(\lambda)\exp(\pi H(z_0 + d, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)) = \chi(\lambda),\ \lambda \in \Lambda \cap N.$$

Putting the two equations together we see that $|\vartheta(z_0 + z)| = |\vartheta(z_0 + d)|$ for all $z \in N$. It follows that, as a function of $z \in N$, $\vartheta(z_0 + z)$ is bounded and therefore constant. Hence $\vartheta(z_0 + z) = \vartheta(z_0)$ for all $z \in N,\ z_0 \in V$. ϑ is then a well-defined theta function for V/N.

Let $\bar{X} = \frac{V/N}{\Lambda/\Lambda \cap N}$, $\pi : X \to \bar{X}$ the projection map. As $\bar{H} \in NS(\bar{X})$ is well-defined and non-degenerate, there is $\bar{L} = L(\bar{H}, \bar{\chi}) \in Pic(\bar{X})$ with

$L = L(H, \chi) = \pi^* \bar{L}$. But then $\pi^* : H^0(\bar{X}, \bar{L}) \to H^0(X, L)$ is an isomorphism which means that any map $\varphi : X \to \mathbf{P}^\nu$ from the linear system $|L|$ must factor through $\bar{X}$:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbf{P}^\nu \\ \pi \downarrow & & \downarrow \mathrm{id} \\ \bar{X} & \xrightarrow{\bar{\varphi}} & \mathbf{P}^\nu \end{array}$$

This contradicts the assumption that L is very ample. □

1.1. The Higher Cohomology Groups $H^i(X, L)$.

Suppose that L is positive definite. Then $h^0(X, L) = d_1 \cdot \ldots \cdot d_g$ where L has type $(d_1, \ldots, d_g)$.

Theorem 1.3. *$H^i(X, L) = 0$ for $i > 0$.*

Proof. We use Kodira Vanishing: If X is a smooth, projective variety and L is an ample line bundle on X, then $H^i(k_X + L) = 0$ for $i > 0$. On a torus, k_X is trivial. □

The case where L is not positive definite is

Theorem 1.4. *Mumford Vanishing: If $H = c_1(L)$ is non-degenerate and has r negative eigenvalues, $H^i(X, L) = 0$ for $i \neq r$ and $H^r(X, L) \neq 0$.*

2. Maps to Tori

Theorem 2.1. *If Y is an abelian variety, X is any variety and $f : X \dashrightarrow Y$ is a rational map such that codim($X \backslash$dom(f),X) ≥ 2, f is regular -ie- dom(f) $= X$.*

As an immediate corollary:

Corollary 2.2. *If X is normal, any rational map $f : X \dashrightarrow Y$ is regular.*

Proof. Let $F : X \times X \dashrightarrow Y$, $F(x_1, x_2) = f(x_1) - f(x_2)$. Clearly dom(f) $\times$ dom(f) $\subset$ dom(F). Then if f is defined at $x \in X$, F is defined at (x, x). We'll prove the converse: If F is defined at (x, x), f is defined at x.

Assume that F is defined at (x_0, x_0). Then there exists some Zariski-open (all open sets will be Zariski-open) $U \subset X \times X$ containing (x_0, x_0) and some $F_U : U \to Y$ such that $F_U(x_1, x_2) = f(x_1) - f(x_2)$ for any $(x_1, x_2) \in U \cap (\mathrm{dom(f)} \times \mathrm{dom(f)})$.

Consider $U' = \{x \in X \mid (x, x_0) \in U\}$. $x_0 \in U'$ so that U' is non-empty and open. But then U' is dense and meets any open set. In particular, U' meets dom(f) -ie- there is some $x_1 \in$ dom(f) such that $(x_1, x_0) \in U$.

Let $V = \{x \in X \mid (x_1, x) \in U\}$. V is an open neighborhood of x_0. Let $\varphi : V \to Y$, $\varphi(x) = f(x_1) - F_U(x_1, x)$. φ extends f:

$$\varphi|_{V \cap \mathrm{dom(f)}} = f$$

since here $\varphi(x) = f(x_1) - f(x_1) + f(x) = f(x)$. This proves the claim.

Now suppose there is some $(x, x) \in X \times X$ so that F is not defined at (x, x). For any variety Z, let $\mathbb{C}(Z)$ be the set of meromorphic functions on

Z, and for any $z \in Z$, let $\mathcal{O}_{Z,z}$ be the local ring of meromorphic functions on Z which are regular at z. We have the following:

$$\begin{array}{ccc} \mathbb{C}(Y) & \xrightarrow{F^*} & \mathbb{C}(X \times X) \\ {\scriptstyle 1\text{-}1}\uparrow & & {\scriptstyle 1\text{-}1}\uparrow \\ \mathcal{O}_{Y,0} & & \mathcal{O}_{X\times X,(x,x)} \end{array}$$

F is not defined at (x,x) if and only if $F^*(\mathcal{O}_{Y,0}) \nsubseteq \mathcal{O}_{X\times X,(x,x)}$. So there is a $v \in \mathcal{O}_{Y,0}$ such that $F^*v = v \circ F$ is not defined at (x,x) -ie- $v \circ F \notin \mathcal{O}_{X\times X,(x,x)}$.

In this case $(x,x) \in (v \circ F)_\infty$, the hypersurface of poles of the meromorphic function $(v \circ F)$. And so F is not defined at any point of $\Delta \cap (v \circ F)_\infty$, a hypersurface in $\Delta \backsimeq X$. This contradicts that f is defined outside codimension two. $\square$

Corollary 2.3. *Any map $F : \boldsymbol{P}^n \dashrightarrow X$, X an abelian variety, is constant.*

Proof. $\mathbf{P}^n$ is smooth and so normal. Then by the theorem just proved, any rational map $F : \mathbf{P}^n \dashrightarrow X$ is regular. To show that F is constant, we may assume $n = 1$ since $\mathbf{P}^1 \subset \mathbf{P}^n$: any map from $\mathbf{P}^n$ would then induce a map from $\mathbf{P}^1$ by restriction.

So consider $F : \mathbf{P}^1 \to X$. On $\mathbf{P}^1$ there are no holomorphic forms: $k_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}(-2)$. However, we have a canonical isomorphism $H^0(X, \Omega^1_X) \backsimeq \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \neq 0$. For any i, $F^*(dz_i) \equiv 0 \Rightarrow dF = 0$ and so F is constant. $\square$

3. Poincaré's Irreducibility Theorem

3.1. **The Norm Map.** As always, (X, L) is a polarized abelian variety. Suppose X contains a proper abelian subvariety Y: $0 \subsetneq Y \subsetneq X$. Let $i : Y \hookrightarrow X$ be the inclusion. As L is ample and Y is a subvariety, i^*L is ample on Y so that (Y, i^*L) is a polarized abelian variety.

Consider the isogeny $\phi_{i^*L} : Y \to \hat{Y}$. $\ker \phi_{i^*L}$ is a finite group and let $e(Y)$ be the exponet of $\ker \phi_{i^*L}$: $\ker \phi_{i^*L} \subset Y[e(Y)]$. The inverse isogeny is $\psi_{i^*L} = e(Y)\phi^{-1}_{i^*L}$.

We define a norm map $N_Y : X \to Y$ as follows: There is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_L} & \hat{X} = \mathrm{Pic}^0(X) \\ {\scriptstyle N_Y}\downarrow & & {\scriptstyle i^*}\downarrow \\ Y & \xleftarrow{\psi_{i^*L}} & \hat{Y} = \mathrm{Pic}^0(X) \end{array}$$

The vertical map $N_Y : X \to Y$ is the norm of Y. Unraveling the definition, we get

$$\begin{array}{ccc} x & \longrightarrow & t_x^*(L) \otimes \check{L} \\ \downarrow & & \downarrow \\ e(Y) \cdot y & \longleftarrow & t_x^*(L)|_Y \otimes \check{L}|_Y \end{array}$$

where

$$t_y^*(L|_Y) \otimes L\check{|}_Y = t_x^*(L)|_Y \otimes L\check{|}_Y \Rightarrow t_y^*(L|_Y) = t_x^*(L)|_Y$$

3.2. **Properties of N_Y.**

(1) $N_Y|_Y(y) = e(Y) \cdot y$
(2) $N_Y^2 = e(Y) \cdot N_Y$

3.3. **Poincaré's Theorem.** Given $0 \subsetneq Y \subsetneq X$ an abelian subvariety, we can define a complimentary abelian subvariety $Z = (\ker N_Y)^0$, the connected component of $\ker N_Y$ containing 0. Then $Z = \{x \in X \mid t_x^*(L)|_Y = L|_Y\}$.

Theorem 3.1. *(Poincaré): $\varphi = "+" : Y \times Z \to X$ given by $(y, z) \mapsto y + z$ is an isogeny.*

Proof. Clearly $dimZ + dimY = dimX$ so we only have to check that $|\ker \varphi| < \infty$. But $\ker \varphi = Y \cap Z = \{y \in Y \mid t_y^*(L) = L_Y\} = k(L_Y) < \infty$ since L is ample. □