

Abelian Varieties, 2/9/06 - Notes taken by Kelly McKinnie

March 6, 2006

0.1 The tensor product of line bundles

We recall the definition of the tensor product of two line bundles on a complex manifold X . Let $L, L' \in \text{Pic}(X) = H^1(X, \mathcal{O}_X^*)$, given by transition functions $L = \{U_{\alpha,\beta}, g_{\alpha,\beta}\}$, $L' = \{U_{\alpha,\beta}, h_{\alpha,\beta}\}$. Then $L \otimes L' := \{U_{\alpha,\beta}, g_{\alpha,\beta}h_{\alpha,\beta}\} \in H^1(X, \mathcal{O}_X^*)$. Last time we defined the first Chern class of a line bundle $c_1 : \text{Pic}(X) \rightarrow H^2(X, \mathbb{Z})$. Note that this map is a group homomorphism with respect to the tensor product of line bundles.

Let $X = V/\Lambda$ be a torus where $V \cong \mathbb{C}^g$ and $\pi : V \rightarrow X$. Let L be a line bundle on X . The pull back of L to V is a trivial line bundle and we have the following diagram,

$$\begin{array}{ccc} & V & \xrightarrow{\pi} & X \\ & \uparrow & & \uparrow \\ V \times \mathbb{C} & \xrightarrow[\cong]{\sim} & \pi^*L & \longrightarrow & L \end{array}$$

This diagram gives us a *factor of automorphy* $\{e_\lambda \in H^0(\mathcal{O}_V^*)\}_{\lambda \in \Lambda}$, that is, $e_\lambda \in H^0(\mathcal{O}_V^*)$ satisfy the compatibility relation

$$e_{\lambda+\lambda'}(z) = e_\lambda(z)e_{\lambda'}(z + \lambda).$$

In particular, the group of line bundles $\text{Pic}(X)$ is canonically identified with $H^1(\Lambda, H^0(\mathcal{O}_V^*))$. The first Chern class, c_1 , is gotten from the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \text{Pic}(X) = H^1(X, \mathcal{O}_X^*) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) & \longrightarrow & \cdots \\ & & \uparrow \simeq & & \parallel & & \\ & & H^1(\Lambda, H^0(\mathcal{O}_V^*)) & \xrightarrow{c_1} & \text{Alt}_{\mathbb{Z}}^2(\Lambda, \mathbb{Z}) & & \end{array}$$

and is given by the following formula

$$c_1(\{e_\lambda\})(\lambda, \mu) = f_\lambda(z) - f_\mu(z) - f_\lambda(z + \mu) + f_\mu(z + \lambda) \in \text{Alt}_{\mathbb{Z}}^2(\Lambda, \mathbb{Z})$$

where the f_λ are chosen so that $e^{2\pi i f_\lambda} = e_\lambda$ and $z \in V$ is an arbitrary point. This formula was proven on 2/7/06.

0.2 The Néron-Severi group

Definition 1. Let X be a complex manifold. The Neron-Severi group of X is defined as $NS(X) := \text{Im}(c_1)$, that is, it consists of all cohomology classes in $H^2(X, \mathbb{Z})$ appearing as first Chern classes of line bundles on X .

We will describe the Neron-Severi group of a torus. Recall if $E = c_1(L)$ where L is a line bundle on a torus X , then $E : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ is a \mathbb{Z} -alternating form and tensoring this with \mathbb{R} we obtain an \mathbb{R} -alternating form $E : V \times V \rightarrow \mathbb{R}$.

Theorem 0.1. *For an \mathbb{R} -alternating form $E : V \times V \rightarrow \mathbb{R}$, the following are equivalent:*

1. $E = c_1(L)$ for a line bundle $L \rightarrow X$.
2. $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$, and $E(ix, iy) = E(x, y)$ for all $x, y \in V$.

Proof. From the short exact sequence of sheaves $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0$ we get the following long exact sequence of cohomology.

$$\cdots \longrightarrow H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \xrightarrow{\alpha} H^2(\mathcal{O}_X) \longrightarrow \cdots$$

By definition $\text{NS}(X)$ is the kernel of α . By extending scalars from \mathbb{Z} to \mathbb{C} , where both \mathbb{Z} and \mathbb{C} are viewed as constant sheaves, we get the following square, which we are going to show is commutative.

$$\begin{array}{ccc} H^2(X, \mathbb{Z}) & \xrightarrow{\alpha} & H^2(\mathcal{O}_X) \\ \downarrow & & \parallel \\ H^2(X, \mathbb{C}) & \xrightarrow{\text{proj}} & H_{\text{Dol}}^{0,2}(X) \end{array} \quad (1)$$

The equality on the right hand side is from Dolbeault's theorem, $H^q(\Omega_X^p) = H_{\text{Dol}}^{p,q}(X)$. The map on the left hand side of (1) is extension of scalars from \mathbb{Z} to \mathbb{C} . To see that (1) commutes, we consider the following morphism of complexes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{A}_X^0 & \xrightarrow{d} & \mathcal{A}_X^1 & \xrightarrow{d} & \cdots \\ & & \downarrow \text{inc} & & \downarrow \pi & \text{proj} & \downarrow \pi & \text{proj} & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{A}_X^{0,0} & \xrightarrow{\bar{\partial}} & \mathcal{A}_X^{0,1} & \xrightarrow{\bar{\partial}} & \cdots \end{array} \quad (2)$$

The top row of (2) is used to compute de Rham cohomology, the bottom row of (2) is used to compute $(0, n)$ -Dolbeault cohomology. This really is a morphism of complexes, that is the squares commute. We can see this for the second square as follows. Let $f \in \mathcal{A}_X^0$.

$$\begin{array}{ccc} f(z) & \xrightarrow{d} & \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \\ \downarrow & & \downarrow \text{proj} \\ f(z) & \xrightarrow{\bar{\partial}} & \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z} \end{array}$$

The morphism of complexes (2) gives us an induced projection map $\text{proj} : H_{\text{dR}}^n(X) \rightarrow H_{\text{Dol}}^{(0,n)}(X)$. By de Rham's theorem, $H^n(X, \mathbb{C}) \cong H_{\text{dR}}^n(X)$, and by Dolbeault's theorem $H^n(\mathcal{O}_X) \cong H_{\text{Dol}}^{(0,n)}(X)$. The induced map $i : H^n(X, \mathbb{C}) \rightarrow H^n(\mathcal{O}_X)$ is really projection from the Hodge decomposition of $H^n(X, \mathbb{C})$ to $H^n(\mathcal{O}_X)$. The following diagram is commutative.

$$\begin{array}{ccc} H^n(X, \mathbb{C}) & \xrightarrow{\sim} & H_{\text{dR}}^n(X) \\ \downarrow i & & \downarrow \text{proj} \\ H^n(\mathcal{O}_X) & \xrightarrow{\sim} & H_{\text{Dol}}^{0,n}(X) \end{array}$$

Applying this for $n = 2$ we get that (1) is commutative. To see that the map on the right is really projection, we first recall the following definitions.

$$H^q(\Omega_X^p) = \bigwedge^p \Omega \otimes \bigwedge^q \overline{\Omega} = H_{\text{Dol}}^{p,q}(X)$$

where $\Omega = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\overline{\Omega} = \text{Hom}_{\mathbb{C}\text{-anti}}(V, \mathbb{C})$ and

$$H_{\text{dR}}^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X).$$

Therefore, $H_{\text{dR}}^2(X, \mathbb{C}) = \bigwedge^2 \Omega \oplus (\Omega \otimes \overline{\Omega}) \oplus \bigwedge^2 \overline{\Omega}$, $H_{\text{Dol}}^{0,2}(X) = \bigwedge^2 \overline{\Omega}$, and the map between them is really projection. We are now almost done. Let $E \in H^2(X, \mathbb{Z})$ with $\alpha(E) = 0$. Then write E in it's decomposition $E = E_1 + E_2 + E_3$ with $E_1 \in \bigwedge^2 \Omega$, $E_2 \in \Omega \otimes \overline{\Omega}$ and $E_3 \in \bigwedge^2 \overline{\Omega}$. By the commutativity of (1) $\alpha(E) = E_3$, and therefore $E_3 = 0$.

Exercise 0.2. Let $E : V \times V \rightarrow \mathbb{C}$ an \mathbb{R} alternating form. Show $E(V \times V) \subseteq \mathbb{R}$ if and only if $E_1 = \overline{E}_3$.

This exercise implies that $E_1 = \overline{E}_3 = 0$, and therefore, $E = E_2$, that is, it is a (1,1) form. The (1,1)-forms are precisely those that satisfy $E(ix, iy) = E(x, y)$. To see this, let $f \otimes g \in \Omega \otimes \overline{\Omega}$, a simple tensor, where $f : V \rightarrow \mathbb{C}$ is \mathbb{C} -linear and $g : V \rightarrow \mathbb{C}$ is \mathbb{C} -anti-linear. We now check that $f \otimes g(ix, iy) = f \otimes g(x, y)$:

$$\begin{aligned} f \otimes g(ix, iy) &= f(ix)g(iy) \\ &= if(x)(-i)g(y) \\ &= f(x)g(y) = f \otimes g(x, y) \end{aligned} \tag{3}$$

Exercise 0.3. Show $\Omega \otimes \overline{\Omega} = \{E : V \times V \rightarrow \mathbb{C} \mid E(ix, iy) = E(x, y)\}$. Just above we proved " \subseteq ". To prove the other inclusion count dimensions.

With this the theorem has been proved. □

Corollary 0.4. There exists a bijection between $\text{NS}(X)$ and the space of Hermitian forms $H : V \times V \rightarrow \mathbb{C}$ such that $\text{Im}H(\Lambda, \Lambda) \subseteq \mathbb{Z}$.

Recall a form is said to be *Hermitian* if it is \mathbb{C} -linear and $\overline{H(x, y)} = H(y, x)$.

Proof. By Theorem 0.1, $\text{NS}(X) = \{E : V \times V \rightarrow \mathbb{R}, \text{alternating} \mid E(\Lambda, \Lambda) \subseteq \mathbb{Z}, \text{ with } E(ix, iy) = E(x, y)\}$. The bijection takes

$$E \mapsto H : V \times V \rightarrow \mathbb{C}$$

where $H(x, y) = E(ix, y) + iE(x, y)$. We need to check that this is a Hermitian form:

$$\begin{aligned} \overline{H(y, x)} &= \overline{E(iy, x) - iE(y, x)} = -E(y, ix) - iE(y, x) \\ &= E(ix, y) + iE(x, y) = H(x, y) \end{aligned}$$

□

0.3 The Appell-Humbert Theorem

The Appell-Humbert Theorem was proved by Appell and Humbert in the 1890's for dimension 2. It was proved by Lefschetz in general. It is the most important theorem so far in the class. Recall our general set up, $L \rightarrow X = V/\Lambda$, a line bundle. From L we get factors of automorphy $\{e_\lambda\}$. We would like to get a **canonical** factor of automorphy representing L , that is, a distinguished representative.

Definition 2. A **semi-character** for $H \in \text{NS}(X)$ is a map $\chi : \Lambda \rightarrow S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ such that

$$\chi(\lambda + \lambda') = \chi(\lambda)\chi(\lambda')e^{\pi i E(\lambda, \lambda')}.$$

Example: $H = 0$; the semi-characters for 0 are $\text{Hom}(\Lambda, S^1)$.

Definition 3. $\mathcal{P}(\Lambda) = \{(H, \chi) \mid H \in \text{NS}(X), \chi \text{ is a semi-character for } H\}$.

$\mathcal{P}(\Lambda)$ is a group with operation $(H_1, \chi_1)(H_2, \chi_2) = (H_1 + H_2, \chi_1\chi_2)$. There exists an exact sequence

$$1 \longrightarrow \text{Hom}(\Lambda, S^1) \longrightarrow \mathcal{P}(\Lambda) \xrightarrow{p} \text{NS}(X).$$

p is the “forget the semi-character” map. It is not clear yet that p is surjective, but it will be true. In fact, we will eventually have $\text{Pic}(X) \cong \mathcal{P}(\Lambda)$, where the map is defined as follows.

$$\mathcal{P}(\Lambda) \xrightarrow{a} \text{Pic}(X) \xrightarrow[\text{can}]{\sim} H^1(\Lambda, H^0(\mathcal{O}_V^*)) \quad (\text{factors of automorphy})$$

$$(H, \chi) \xrightarrow{a} L(H, \chi)$$

where $a(H, \chi) = L(H, \chi)$ is the line bundle given by the factors of automorphy

$$a(H, \chi) = \{a_\lambda(z) = \chi(\lambda)e^{\pi H(z, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)}\}_{\lambda \in H^0(\mathcal{O}_V^*)}.$$

We need to show that this definition makes $\{a_\lambda\}$ into a factor of automorphy, that is, we need to show that $a_{\lambda+\lambda'}(z) = a_\lambda(z)a_{\lambda'}(z + \lambda)$:

$$\begin{aligned} a_{\lambda+\lambda'}(z) &= \chi(\lambda + \lambda')e^{\pi H(z, \lambda+\lambda') + \frac{\pi}{2}H(\lambda+\lambda', \lambda+\lambda')} \\ &= \chi(\lambda)\chi(\lambda')e^{\pi iE(\lambda, \lambda')}e^{\pi H(z, \lambda') + \pi H(z, \lambda) + \frac{\pi}{2}[H(\lambda, \lambda) + H(\lambda', \lambda') + H(\lambda, \lambda') + H(\lambda', \lambda)]} \\ &= \left(\chi(\lambda)e^{\pi H(z, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)}\right) \left(\chi(\lambda')e^{\pi H(z, \lambda') + \frac{\pi}{2}H(\lambda', \lambda')}e^{\pi iE(\lambda, \lambda') + \pi \text{Re}(H(\lambda, \lambda'))}\right) \\ &= \left(\chi(\lambda)e^{\pi H(z, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)}\right) \left(\chi(\lambda')e^{\pi H(z+\lambda, \lambda') + \frac{\pi}{2}H(\lambda', \lambda')}\right) \\ &= a_\lambda(z)a_{\lambda'}(z + \lambda) \end{aligned}$$

The 4-th equality follows from the relation $H(x, y) = E(ix, y) + iE(x, y)$.

The Appell-Humbert Theorem says that $a : \mathcal{P}(\Lambda) \rightarrow \text{Pic}(X)$ is an isomorphism. We will prove this next time. This gives a geometric description of what all line bundles look like. Because, as a total space, we have:

$$L(H, \chi) = V \times \mathbb{C} / \sim$$

where $(z, t) \sim (z + \lambda, a_\lambda(z)t)$.

Proposition 0.5. *The diagram*

$$\begin{array}{ccc} \mathcal{P}(\Lambda) & \xrightarrow{a} & \text{Pic}(X) \\ & \searrow p & \swarrow c_1 \\ & \text{NS}(X) & \end{array}$$

is commutative where p is the “forget the semi-character” map. In particular, $c_1(L(H, \chi)) = H$.